$x - y$; that is, the theorem is not true for $n - 1$. This descent argument, repeated a finite number of times, requires that the theorem be not true when $n = 2$; in other words, that $x^2 - y^2$ be not divisible by $x - y$. This furnishes the contradiction needed to complete the proof by reductio ad absurdum. The descent is not an infinite descent but rather a finite descent to the special verified case of the theorem. It is really a proof by mathematical induction turned around. Any proof by mathematical induction can be put in this descent form. The proof is well characterized by the description which Fermat gave of his method in connection with the theorem that every prime of the form $4k + 1$ equals the sum of two squares. But Fermat's proof of this theorem does not involve an argument from n to $n - 1$. Nor is the descent in this case one from a prime number to the next smaller prime number. The essential feature of his method is not any such regular descent as that given by the argument from n to $n - 1$. It is rather a descent from an integer n involved in the theorem to some smaller integer; just how much smaller may not be known. This is very well illustrated by the proof that $x^4 + y^4 = z^4$ has no solution in integers different from zero. The proof in outline is this: If $x^4 + y^4 = z^4$ has a solution in such integers, so has $x^4 + y^4 = z^2$; if there exists a solution (x_1, y_1, z_1) of this equation, there exists another solution (x_2, y_2, z_2) in which z_2 is an integer less than z_1; and by the same argument another solution (x_3, y_3, z_3) in which z_3 is less than z_2; and so forth ad infinitum. The contradiction neede for the completion of the argument by reductio ad absurdum is furnished by the fact that there is not an infinite number of positive integers less than a given one. But the contradiction may just as well be obtained from the special case $x^4 + y^4 = 1^2$, which by inspection has no solution in integers different from zero. The descent argument that $z_2 < z_1$ holds without exception. But in the proof that $x^n - y^n$ is divisible by $x - y$ the descent argument fails when $n = 1$. The fact that it fails makes the proof a finite descent proof rather than an infinite descent proof. But it should be said in this connection that the proof that $x^4 + y^4 = z^2$ has no integral solution may be thought of as a finite descent to the special case $x^4 + y^4 = 1^2$ mentioned above.

The proof that $x^4 + y^4 = z^4$ has no integral solution may also be put in this form.[1] Let it be assumed that $x^4 + y^4 = z^2$ has one or more solutions and let (x_1, y_1, z_1) be a solution in which z_1 is the smallest possible. Then the descent argument proves that there is a solution (x_2, y_2, z_2) in which z_2 is less than z_1. The required contradiction is furnished by the fact that there is thus proved to be a smaller than the smallest. In any true infinite descent proof, that is in a proof in which the descent argument holds without exception, the contradiction needed for the reductio ad absurdum argument may be obtained in any one of the three ways mentioned, namely from the fact that there is not an infinite number of integers less than a given integer, or from the descent to a special verified case of the theorem, or from the fact that there is not a smaller than the smallest. But when the descent argument does not hold without exception, as

[1] See H. WEBER and J. WELLSTEIN, *l. c.*, p. 284.

in the proof that $x^n - y^n$ is divisible by $x - y$, the contradiction must be obtained from the descent to a special verified case of the theorem. It is this finite descent proof which is mathematical induction turned around. In thinking of these two types of descent proof one wonders if there is not another type of ascent proof than the regular ascent from n to $n + 1$ of mathematical induction. The well-known proof, due to Euclid, that there is no greatest prime is one such. The argument is that if p is any prime, the number $N = p! + 1$ is a greater prime or else one of its prime factors is; that is, if p is any prime, there is a greater prime.

It is hard to account for Fermat's trouble in applying his method to positive or affirmative propositions. His method is to assume the opposite of what he wants to prove and to show that the assumption leads to a contradiction. Sometimes the statement of a proof by reductio ad absurdum is awkward, but it may be just as awkward for a negative proposition as for an affirmative one. His difficulty reminds one of the student who can prove by mathematical induction that $x^{2n} - y^{2n}$ is divisible by $x + y$ but is at a loss to know how to use the same method to prove that $x^{2n} + y^{2n}$ is not divisible by $x + y$.

Fermat, in his letter to Carcavi, says or implies that his method of infinite descent is applicable to the following propositions, some of them having already been mentioned in this paper:

1. There is no number of the form $3k - 1$ which is also of the form $x^2 + 3y^2$.
2. There is no right triangle whose sides are integers and whose area equals a square number.
3. Every prime of the form $4k + 1$ is equal to the sum of two squares.
4. Every number is either a square or the sum of two, three or four squares.
5. $x^2 - ay^2 = 1$ has an infinite number of integral solutions if a is not a square.
6. There is no integral solution of $x^3 + y^3 = z^3$.
7. There is one and only one integral solution of $x^2 + 2 = y^3$, namely, $x = 5$, $y = 3$.
8. There are two and only two integral solutions of $x^2 + 4 = y^3$, namely, $x = 2$, $y = 2$ and $x = 11$, $y = 5$.
9. Every number of the form $2^{2^n} + 1$ is a prime number.
10. There are only two integral solutions of $(2x^2 - 1)^2 = 2y^2 - 1$, namely, $x = 1$, $y = 1$ and $x = 2$, $y = 5$.

Fermat does not say explicitly that he succeeded in proving all of these propositions but his language gives one that impression. He does say that he proved Nos. 1, 2, 3, 4, 5, 10. Of Nos. 6, 7, 8, 9 he says: "I afterwards considered certain propositions which, although negative, present very great difficulty, the method of applying the descent being altogether different from that of the preceding cases." Of No. 9 he says further: "This is a very subtle and very ingenious research and, although it is stated affirmatively, it is negative, since to say that a number is prime is the same as to say that it cannot be divided by any number."

This Proposition No. 9 is not true. For when $n = 5$ the number

$$2^{2^n} + 1 = 4{,}294{,}967{,}297 = 641 \cdot 6{,}700{,}417.^1$$

It is often stated that Fermat did not claim to have proved this theorem. This statement is based on the fact that in a letter to Pascal[2] in August, 1654, he says that, although he is convinced of its truth, he has not been able to find a valid demonstration; and that in a letter to Digby[3] in June, 1658, he says that he is seeking a demonstration of this beautiful theorem. But the quotation given above is from the 1659 letter to Carcavi and in it Fermat seems to say that he has proved it by the method of infinite descent.

For a brief account of Fermat's method of infinite descent, especially of its relation to the solution of Diophantine equations, the reader is referred to R. D. CARMICHAEL'S "*Diophantine Analysis*," pp. 14–21. The reader is also referred to the supplement of Sir THOMAS L. HEATH'S "*Diophantus of Alexandria*," Cambridge University Press, 2d edition, 1910, which contains a number of references to the method of infinite descent.

THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS.

By JOSEF NYBERG, Hyde Park High School, Chicago, Ill.

In a previous paper[4] I showed how the study of the line in analytic geometry could be presented from the viewpoint of functions and relations between variables. The present article explains how logarithms and exponential relations are introduced on the same basis.

I begin the work by solving some problem leading to an exponential relation between two variables. Most such problems taken from physics or chemistry lead to a negative exponent, which is undesirable at the beginning of the work. However, a suitable problem is that of determining the number N of bacteria in a solution at a time t, knowing the number N_0 present at $t = 0$ and knowing the rate of increase. The conditions are similar to those for finding the length of a rod with increasing temperature, a problem studied under linear functions, but different in that the coefficient of expansion of the rod is independent of the temperature while the bacteria increase at a rate dependent upon t. If the bacteria increase r per cent. in a unit of time, then

$$N = N_0\left(1 + \frac{r}{100}\right)^t.$$

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[1] This was proved by Euler in 1732. For this and other comments on the theorem see W. W. R. Ball, *Mathematical Recreations and Essays*, 5th edition, p. 39. Also see R. C. Archibald's *Remarks on "Klein's famous problems of elementary geometry,"* this MONTHLY, Vol. 21, p. 248.

[2] *Oeuvres de Fermat*, Vol. 2, p. 307.

[3] *Ibid.*, p. 402.

[4] "The Linear Function and the Line," in this MONTHLY, Nov., 1917, page 406. This was the third of a series of related papers of which the other two are:
"The Unification of Freshman Mathematics," April, 1916, page 101;
"The Presentation of the Notion of Function," Sept., 1917, p. 309.

The next work consists in drawing carefully a graph of $y = a^x$ taking $a = 2$ as the most convenient. We find points on the curve by geometrical constructions using the line $y = ax$ and interpolating points by using the lines $y = \sqrt{a}x$, $y = \sqrt[4]{a}x$, etc. (remembering that the logarithms increase in arithmetic progression when the numbers increase in geometric progression). At once curves like $y = y_0a^x$ can be sketched by a proper multiplication of the ordinates, and $y = y_0a^{nx}$ by a multiplication of the abscissas. Since the latter curve is the same as $y = y_0b^x$ where $b = a^n$, any exponential curve can be sketched from the one fundamental one. Moreover a satisfactory value of n can be found from the graph of $y = a^x$ inasmuch as n is the abscissa of the point whose ordinate is b. We call n the logarithm of b to the base a.

Having seen how the exponential function arises, and how it differs from the linear function in its rate of increase, graphed it, and introduced the word logarithm, we are ready to study the properties of the function. It is at once evident that negative numbers have no real logarithms since the curve by our construction can never get below the x axis, that the logarithm of the base is 1, that $\log 1 = 0$, and that numbers less than 1 have negative logarithms. Progressively we prove the fundamental relations for $\log (N_1/N_2)$, $\log N_1N_2$, and $\log_a N = \log_b N \cdot \log_a b$. For exercises in drill I use numbers whose logarithms can be taken from the graph which the student has drawn, or else use integers less than 40 whose logarithms I have written on the blackboard. The student computes log 3, log .04, etc., from the properties of the function. At this point the student also solves such exponential equations as $p^x \cdot q^{x+2} = r$ (p, q, r, < 40). The difficulties in this equation which Mr. Simpson[1] pointed out can never arise under this method of treatment because negative logarithms are no less common than positive ones, and because the notion of the characteristic and mantissa have not yet been introduced. Mantissas and characteristics are in fact unnecessary until we begin using tables.

Before proceeding to the use of tables I like to explain at this point the significance of the constant e as a base. The fact that $y = e^x$ has at any point a slope equal to the ordinate at that point does not appear to be unusually significant, not even to a student of calculus; nor can the student appreciate its significance as the limiting value of a certain series unless we show him *how* and *why* this particular series is *unavoidable* in the problems. This can be done by returning to the original problem of the bacteria which led to the relation

$$N = N_0\left(1 + \frac{r}{100}\right)^t,$$

when the rate of increase is r per cent. per unit of time. If the unit of time is cut in two, then

$$N = N_0\left(1 + \frac{r}{200}\right)^{2t};$$

[1] "Relating to the Teaching of Logarithms," in this MONTHLY, Sept., 1916, page 264.

and if divided into q parts, then

$$N = N_0\left(1 + \frac{r}{100q}\right)^{qt}.$$

We wish to derive the expression for N when q is increased indefinitely (*i. e.*, when the unit is decreased in size). Put $r/100q = 1/u$ so that

$$N = N_0\left(1 + \frac{1}{u}\right)^{urt/100}.$$

We may then show that $\lim_{u=\infty}(1 + 1/u)^u$ is a definite constant, which we denote by e. This work may be done with as much rigor as the individual teacher prefers. Thereafter similar problems are assigned from physics, as for example, the amount of light coming through a pane of glass, or problems from chemistry on the decomposition of elements. r is negative for most of these problems, but this fact is not now a disturbing element to the student. These problems also have the advantage that the constants y_0 and n in the relation $y = y_0e^{nx}$ have a definite physical association.

Not until the previous theory is clear do I consider logarithms in their application to computing. There are two new ideas that need to be impressed on the student: (1) the conventions of the computer of dividing a logarithm, irrespective of whether it is positive or negative, into two parts, an integral characteristic and a decimal mantissa; (2) the conventions of the printer used in abbreviating the work of printing the table. Considering also the definition previously given, we are dealing with a logarithm from three distinct aspects: first, what it actually is, as log .6974 = − .15658; second, what the printer puts into the table, as 84342; third, what the computer puts on his paper, as .84342 − 1.

The textbooks do not clearly differentiate these three steps, and yet every error of the beginner is due to the confusion of these notions. It alone is responsible for the error that Mr. Simpson mentions; it leads to the usual errors in handling numbers less than unity, and to errors in finding the antilogarithm. The first point can be emphasized by having the student check his logarithms (if he is working the customary exercises of finding the logarithms of an assigned list of numbers) from his graph. Under the second item I frequently resort to the homely method of making the student put a decimal point before each mantissa in some column of his table. This cures him permanently of any inclination to write .08434 instead of .8434 − 1 for log .6974, or of writing 212.7 instead of 3.2127 for log 1632. The abbreviation of the printer's work is also the justification of 10 as a base in computing, whereas we use e as a base for exponential relations arising in problems other than computing. This the student will readily see if he computes $\log_e 1.234$, $\log_e 12.34$, etc. In fact there is scarcely a better way for learning the rules for handling logarithms than by comparing the similar rules for e as the base. Under the third idea of what the computer puts on his paper, the teacher can explain the advantage of cologarithms, the necessity of writing 2.4133 − 3 instead of .4133 − 1 for log .259 in finding its cube root, and similar matters.

This tripartite analysis of a logarithm is more powerful than may appear at a first reading. We learn by contrast; and the value of this particular explanation lies in seeing logarithms from the viewpoints of the computer, the printer, and the graph. Most teachers are inclined to believe that the best instruction consists in presenting to the student only one idea at a time. Psychologically, however, it is equally sound to keep together two closely related ideas, and teach by contrast and comparisons. Ordinarily logarithms are taught solely for their use in computing; later in analytic geometry the student draws perhaps a few exponential curves; and then in calculus he learns about the base e. The introduction of both bases simultaneously impresses the significant facts about each; the mind is put in a receptive mood when it sees the necessity of the rules. The writer believes the present paper affords a good illustration of how various parts hitherto separated in the curriculum by years can be taught sequentially when organized on the basis of functions. In addition to unity of subject matter, we also obtain an unusual degree of coherence. Each day's work is intimately linked with the preceding and prepares for the following.

NOTE ON LAGRANGE'S LIKE-PRODUCING QUADRINOMIAL.[1]

By Sir THOMAS MUIR, South Africa.

1. From an identity originally due to Lagrange we know that the product

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)(y_1^2 + bcy_2^2 + cay_3^2 + aby_4^2)$$

can be expressed in the same form as either factor, namely, in the form

$$P_1^2 + bcP_2^2 + caP_3^2 + abP_4^2.$$

For the case, however, where the second factor is the same as the first, the formula is nugatory, all that it then tells us being that $(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^2$ is equal to itself. It seems therefore not a little interesting to know that without introducing any new principle we can show that the product of the *cube* of the first factor by the second factor can be simply expressed in the particular form in question.

2. We recall in the first place that

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^2 = \begin{vmatrix} x_1 & bcx_2 & cax_3 & abx_4 \\ -x_2 & x_1 & -ax_4 & ax_3 \\ -x_3 & bx_4 & x_1 & -bx_2 \\ -x_4 & -cx_3 & cx_2 & x_1 \end{vmatrix},$$

and that each primary minor of this determinant contains the square root of the

[1] Part of the object of this paper is to throw a little light on the subject of Problem 489 of this MONTHLY for Nov. 1917.

determinant as a factor, the adjugate of the determinant being

$$\begin{vmatrix} x_1Q & x_2Q & x_3Q & x_4Q \\ -bcx_2Q & x_1Q & -bx_4Q & cx_3Q \\ -cax_3Q & ax_4Q & x_1Q & -cx_2Q \\ -abx_4Q & -ax_3Q & bx_2Q & x_1Q \end{vmatrix}$$

if we put Q for $x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2$.

3. From this it follows that if we take any three columns of the said determinant along with *any fourth column whatever* to form a new determinant, the latter must have Q for a factor. A special interest attaches to the choosing of y_1, y_2, y_3, y_4 for the fourth column: for the new determinants and their factors then are

$$\begin{vmatrix} y_1 & bcx_2 & cax_3 & abx_4 \\ y_2 & x_1 & -ax_4 & ax_3 \\ y_3 & bx_4 & x_1 & -bx_2 \\ y_4 & -cx_3 & cx_2 & x_1 \end{vmatrix} = Q(x_1y_1 - bcx_2y_2 - cax_3y_3 - abx_4y_4),$$

$$\begin{vmatrix} y_1 & x_1 & cax_3 & abx_4 \\ y_2 & -x_2 & -ax_4 & ax_3 \\ y_3 & -x_3 & x_1 & -bx_2 \\ y_4 & -x_4 & cx_2 & x_1 \end{vmatrix} = Q(-x_2y_1 - x_1y_2 - ax_4y_3 + ax_3y_4),$$

$$\begin{vmatrix} y_1 & x_1 & bcx_2 & abx_4 \\ y_2 & -x_2 & x_1 & ax_3 \\ y_3 & -x_3 & bx_4 & -bx^2 \\ y_4 & -x_4 & -cx_3 & x^1 \end{vmatrix} = Q(x_3y_1 - bx_4y_2 + x_1y_3 + bx_2y_4),$$

$$\begin{vmatrix} y_1 & x_1 & bcx_2 & cax_3 \\ y_2 & -x_2 & x_1 & -ax_4 \\ y_3 & -x_3 & bx_4 & x_1 \\ y_4 & -x_4 & -cx_3 & cx_2 \end{vmatrix} = Q(-x_4y_1 - cx_3y_2 + cx_2y_3 - x_1y_4),$$

where the cofactors of Q turn out to be values of P_1, P_2, P_3, P_4 in § 1 as given by Lagrange's rule. The consequence is that if in the equality of § 1 we substitute for each P the quotient of the appropriate determinant by Q, we obtain the following theorem:

If by $|rstu|$ *be denoted the determinant whose columns are the* rth, sth, tth, uth *columns of the array*

$$\begin{matrix} y_1 & x_1 & bcx_2 & cax_3 & abx_4 \\ y_2 & -x_2 & x_1 & -ax_4 & ax_3 \\ y_3 & -x_3 & bx_4 & x_1 & -bx_2 \\ y_4 & -x_4 & -cx_3 & cx_2 & x_1 \end{matrix}$$

then

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)^3 \cdot (y_1^2 + bcy_2^2 + cay_3^2 + aby_4^2)$$
$$= |1345|^2 + bc|1245|^2 + ca|1235|^2 + ab|1234|^2.$$

4. From Lagrange's result it of course follows that

$$(x_1^2 + bcx_2^2 + cax_3^2 + abx_4^2)(y_1^2 + bcy_2^2 + \cdots)(z_1^2 + bcz_2^2 + \cdots)$$

can be expressed in the same form as any one of the factors, say in the form

$$T_1^2 + bcT_2^2 + caT_3^2 + abT_4^2,$$

and there thus arises the problem of finding the T's in terms of the x's, y's, z's and a, b, c. The simplest way of writing the four expressions obtained is by means of the notation for bipartite functions, according to which

$$\begin{array}{ccc|c} x & y & z & \\ \hline a & b & c & \xi \\ d & e & f & \eta \\ g & h & i & \zeta \end{array} \quad \text{stands for } ax\xi + by\xi + cz\xi + dx\eta + ey\eta + fz\eta + gx\zeta + hy\zeta + iz\zeta,$$

each element of the square array being taken along with the outside element in the same column and at the same time along with the outside element in the same row. We then have

$$T_1 = \begin{array}{cccc|c} x_1 & x_2 & x_3 & x_4 & \\ \hline y_1 & bcy_2 & cay_4 & aby_4 & z_1 \\ -y_2 & y_1 & -ay_4 & ay_3 & bcz_2 \\ -y_3 & by_4 & y_1 & -by_2 & caz_3 \\ -y_4 & -cy_3 & cy_2 & y_1 & abz_4 \end{array}$$

and the other T's differing from it merely in having for their outside column a different row of the determinant equivalent for $(z_1^2 + bcz_2^2 + caz_3^2 + abz_4^2)^2$.

QUESTIONS AND DISCUSSIONS.

SEND ALL COMMUNICATIONS TO U. G. MITCHELL, University of Kansas, Lawrence.

DISCUSSIONS.

We believe that those of our readers who have followed the discussions of certain solutions of cubic equations as published in the February and June numbers of the MONTHLY will be especially interested in the method given below for approximating nearly equal roots of a cubic.

I. Relating to Approximations to Nearly Equal Roots of a Cubic Equation.

By Paul Capron, U. S. Naval Academy, Annapolis.

If $y = f(x) \equiv x^3 - 3a^2x + 2ma^3$, the equation $y = 0$ has three real roots when $m^2 \not> 1$, has a double root $\pm a$ when $m = \pm 1$, and has two roots nearly equal to $\pm a$ when m is nearly equal to ± 1. The discussion that follows is confined to the case in which the nearly equal roots are positive; it is applicable to negative roots if the signs of both m and a are reversed, or if the equation is replaced by $\bar{y} \equiv -f(-x) = 0$.

The method of arriving at the approximation is as follows:

Let the larger of the two roots be x_1, the smaller x_2, and let the graph of $f(x)$ be a curve s of which the points P_1, P_2 correspond to the roots x_1, x_2. As the roots are nearly equal, P_1 and P_2 are joined by a relatively short segment of s, which includes the minimum point, L, and does not include the inflection. (These conditions define the restriction implied by the words "nearly equal.") Suppose a quadratic parabola s', having the equation $y_1 = f_1(x)$, to intersect the cubic parabola s three times at L, three times at infinity in the direction of the y axis, and consequently nowhere else. Then s' crosses s at L, and as it is not as steep as s for large values of x, keeps to the right of s where $x > a$, to the left where $x < a$. (At L, $x = a$.)

Let the roots determined by $y_1 = 0$ be x_1', x_2', of which x_1' is the larger. Then $x_1' > x_1$, $x_2' > x_2$. P_1' and P_2' being the points of s' corresponding to the roots x_1', x_2', let ordinates to the cubic s be drawn through P_1' and P_2', meeting s at Q_1 and Q_2. Now let tangents to s at Q_1 and Q_2 meet the x-axis at P_1'' and P_2'', of which the abscissas are x_1'' and x_2''. As the curvature of s does not change sign from P_1 to P_2, it is evident that $x_1'' > x_1$, $x_2'' < x_2$ and also that $f(a) < 0, f(x_1') > 0, f(x_1'') > 0, f(x_2') < 0, f(x_2'') > 0$.

The values $(x_1' - x_1'')$ and $(x_2' - x_2'')$ to be subtracted from x_1' and x_2', are the quotients of the ordinates of s at Q_1 and Q_2 by the slopes of s at these points; it will be found convenient to use a smaller divisor than the actual slope at Q_1, and a (numerically) larger one at Q_2, with the result that, the corrected roots thus found being x_1''' and x_2''', $f(x_1''') < 0$ and $f(x_2''') < 0$; that is, P_1''' and P_2''' being the points of the x-axis having the abscissas x_1''' and x_2''', P_1''' is enough to the left of P_1'' to be to the left of P_1 and P_2''' is enough to the right of P_2'' to be to the right of P_2.

Thus, of the successive approximations to x_1, viz., a, x_1', x_1'', x_1''', the first and last are too small, the others too large. In each case, x''' is the more convenient of the closer approximations, and the error in using it is less than $|x'' - x'''|$. In case the error is undesirably large, a closer approximation, but without limits of error, is obtained by finding the ratio (practically the same in all cases) in which the segment $P''P'''$ is divided by the chord joining the points of s that have the abscissas x'' and x'''.

The resulting approximation may be formulated:

If in the equation $x^3 - 3a^2x + 2ma^3 = 0$, m is nearly equal to 1, let $\frac{1}{9}(1 - m) = \mu$, $\sqrt[2]{\frac{2}{3}(1 - m)} = \lambda$; then the nearly equal roots are

$$x = a(1 \pm \lambda - \mu)$$

with an error less than

$$a\mu \frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda},$$

the larger approximation being too small, the smaller too large. If this limit of error is unsatisfactory, much closer results may be obtained by applying five-sixths of its value as a correction to increase the larger approximation or decrease the smaller approximation.

As an illustration of the method, consider the equation

$$8x^3 - 36x^2 + 39x + 3 = 0.$$

When the roots are diminished by 3/2, this equation is transformed into $x^3 - \frac{15}{8}x + \frac{15}{16} = 0$. Here $a = \frac{1}{4}\sqrt{10}$, $m = \frac{3}{10}\sqrt{10}$, $\lambda = \sqrt[2]{\frac{2}{3}(1 - m)} = 0.184962$, $\mu = \frac{1}{9}(1 - m) = 0.005702$;

$$x = \frac{\sqrt{10}}{4}(1 \pm 0.184962 - 0.005702) = 0.93229 \text{ or } 0.63984.$$

Adding 1.5, we have 2.43229 (too small) and 2.13984 (too large).

The errors are $< a\mu \cdot \frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda} = \frac{\sqrt{10}}{4} \times 0.0057 \times \frac{0.0925}{1 \pm 0.0925} < 0.0005$.

If the results are not as close as desired, we may compute the errors more carefully (they are 0.00038 + for the larger root, 0.00046 − for the smaller), and, increasing the larger root by $\frac{5}{6} \times 0.00038 = 0.00032$, and decreasing the smaller by $\frac{5}{6} \times 0.00046 = 0.00038$, we may say that the roots are probably 2.43261 and 2.13946. (To six decimals the roots are actually 2.432617 and 2.139465.)

In Wentworth's College Algebra, the equation $x^3 - 515x^2 + 1155x - 649 = 0$ is shown to have two very nearly equal roots, approximately 1.1230914 and 1.1270002. If we multiply the roots of this equation by 3 and diminish the roots of the transformed equation by 515 we have

$$x^3 - 785280x - 267845848 = 0.$$

Here

$$a = 8\sqrt{4090}, \quad m = -66961462\sqrt{4090} \div (65440)^2.$$

$$\sqrt{4090} = 64(1 - p),$$

where

$$p = 0.0007326902925323856,$$

$$(1 + m) = (65440)^{-2}(-3139968 + 4285533568p) = (65440)^{-2}(0.8435952782),$$

$$(1 + m) = 1.969762360 \times 10^{-10}; \quad \mu = \tfrac{1}{9}(1 + m) = 2.18862384 \times 10^{-11},$$

$$\lambda = \sqrt{\tfrac{2}{3}(1+m)} = (65440)^{-1}\sqrt{0.5623968521}$$

$$= 0.7499312316 \div 65440 = 1.145982933 \times 10^{-5},$$

$$x = -a(1 \pm \lambda - \mu) = -512(1-p)(1 \pm \lambda - \mu)$$
$$= -512 + 512p + 512(1-p)(\mp\lambda + \mu),$$

$$x + 515 = 3 + 0.3751374297765814 + 511.6248625702234186(\mp\lambda + \mu),$$

$$\tfrac{1}{3}(x + 515) = 1.123091428324 \text{ or } 1.127000184062.$$

The error is less than

$$\tfrac{1}{3}a\mu\frac{\frac{1}{2}\lambda}{1 \pm \frac{1}{2}\lambda} < \tfrac{1}{3} \times 512(1-p) \times 2.19 \times 10^{-11} \times \frac{5.73 \times 10^{-6}}{1 \pm 5.73 \times 10^{-6}} < 2.14 \times 10^{-14}.$$

In this case, the approximation might safely have been carried one or two places further.

The derivation of the formula follows:

Given $y \equiv f(x) \equiv (x^3 - 3a^2x + 2ma^3)$; transforming the origin of coördinates to the minimum point, L, $(a, 2(m-1)a^3)$, we obtain $y_1 = x_1^3 \pm 3ax_1^2$. The desired quadratic approximation to y_1 near L is $y_2 = 3ax_1^2$. Transformed to the original system of coördinates, the latter becomes $y_3 = 3ax^2 - 6a^2x + (1+2m)a^3$. Solving $y_3 = 0$ we find $x_1' = a(1 + \sqrt{\tfrac{2}{3}(1-m)})$, $x_2' = a(1 - \sqrt{\tfrac{2}{3}(1-m)})$.

Substituting in $y = f(x) \equiv x^3 - 3a^2x + 2ma^3$, we obtain $f(x_1') = a^3[\tfrac{2}{3}(1-m)]^{3/2}$, $f(x_2') = -a^3[\tfrac{2}{3}(1-m)^{3/2}]$.

The derivative is $f'(x) \equiv 3(a^2 - x^2)$; $f'(x_1') = 2a^2(1 - m + \sqrt{6(1-m)}) > 0$, $f'(x_2') = 2a^2(1 - m - \sqrt{6(1-m)}) < 0$.

$$x_1' - x_1'' = \frac{f(x_1')}{f'(x_1')} = a\frac{[\tfrac{2}{3}(1-m)]^{3/2}}{2[1 - m + 3\sqrt{\tfrac{2}{3}(1-m)}]} = \frac{a}{9} \cdot \frac{1-m}{1 + \frac{1}{2}\sqrt{\tfrac{2}{3}(1-m)}};$$

similarly

$$x_2' - x_2'' = \frac{a}{9} \cdot \frac{1-m}{1 - \frac{1}{2}\sqrt{\tfrac{2}{3}(1-m)}},$$

$$x_1'' = a\left[1 + \sqrt{\tfrac{2}{3}(1-m)} - \frac{1}{9}\frac{1-m}{1 + \frac{1}{2}\sqrt{\tfrac{2}{3}(1-m)}}\right];$$

$$x_2'' = a\left[1 - \sqrt{\tfrac{2}{3}(1-m)} - \frac{1}{9} \cdot \frac{1-m}{1 - \frac{1}{2}\sqrt{\tfrac{2}{3}(1-m)}}\right].$$

Neglecting the radical in the denominator, we have:

$$x_1''' = a[1 + \sqrt{\tfrac{2}{3}(1-m)} - \tfrac{1}{9}(1-m)] < x_1'';$$

$$x_2''' = a[1 - \sqrt{\tfrac{2}{3}(1-m)} - \tfrac{1}{9}(1-m)] > x_2''.$$

Also

$$f(x_1''') = \frac{a^3}{27}(1-m)^2[-5+\sqrt{\tfrac{2}{3}(1-m)}-\tfrac{1}{27}(1-m)] < 0.$$

$$f(x_2''') = \frac{a^3}{27}(1-m)^2[-5-\sqrt{\tfrac{2}{3}(1-m)}-\tfrac{1}{27}(1-m)] < 0.$$

Therefore the errors are less, respectively, than

$$x_1'' - x_1''' = \frac{a}{9}(1-m)\left[1-\frac{1}{1+\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}\right] = \frac{a}{9}(1-m)\cdot\frac{\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}{1+\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}},$$

$$x_2''' - x_2'' = \frac{a}{9}(1-m)\left[\frac{1}{1-\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}-1\right] = \frac{a}{9}(1-m)\cdot\frac{\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}{1-\frac{1}{2}\sqrt{\frac{2}{3}(1-m)}}.$$

Hence the rule given earlier for nearly equal positive roots. For nearly equal negative roots (m nearly -1), if $\frac{1}{9}(1+m)=\mu$, $\sqrt{\frac{2}{3}(1+m)}=\lambda$, $x=-a(1\pm\lambda-\mu)$, with an error less than $a\mu\cdot(\frac{1}{2}\lambda/(1\pm\frac{1}{2}\lambda))$, the numerically larger root being too small numerically, the other too large numerically.

Whether the approximate roots are positive or negative, the one further from the inflection is too small, the one nearer the inflection too large; in other words, both are on the portion of the x-axis intercepted by the adjacent arch of the graph of $y=0$.

If we let $\frac{1}{2}\mu = z$, we have $f(x_1''') = -12a^3z^2(5-2\sqrt{3z}-6z)$ as already seen, and further, $f(x_1'') = 12a^3z^2(1-3z)^{-2}(1-\frac{38}{8}z+21z^2+8z(1-2z)\sqrt{3z})$; $f(x_2''')$ and $f(x_2'')$ differ only in the sign of $\sqrt{3z}$. Consequently

$$-f(x_1''') : f(x_1'') = 5:1$$

very nearly, for either root. Approximating by means of the chord of the graph, we thus obtain a closer result, for which, however, no limits of error are fixed. If the limit of error of the earlier approximation is

$$a\mu\frac{\frac{1}{2}\lambda}{1\pm\frac{1}{2}\lambda} = E_1, E_2$$

(E_1 taking the upper sign, E_2 the lower)

$$x = a(1\pm\lambda-\mu\pm\tfrac{5}{6}E_{1,2})$$

if the roots are positive;

$$x = -a(1\pm\lambda-\mu\pm\tfrac{5}{6}E_{1,2})$$

if the roots are negative.

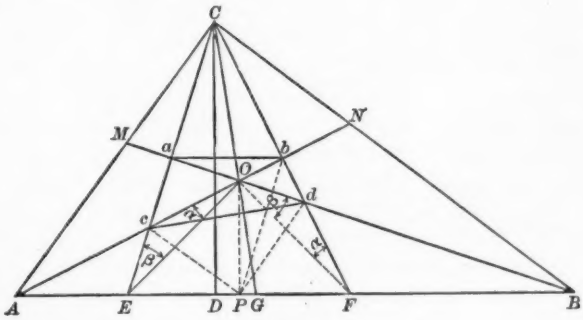
The chief advantages of the approximation are that it gives an easy means of locating two roots, and of separating them if they are very nearly equal, and in the latter case gives an approximate solution more convenient than Horner's method and more accurate than the trigonometric solution. The formulas for

the trigonometric solution are $\cos 3\theta = -m$, $x = 2a\cos\theta$; if $|m|$ is nearly unity, the value of θ is not very accurately determined. The first of the examples cited above is near the border-line; five-place tables will give nearly as accurate results by the trigonometric method as by the approximation, but with more labor. [The roots by this method are -0.0721, 2.4326, 2.1394.]

II. Relating to Some Relations in a Right-angled Triangle.

By Albert Babbitt, University of Nebraska, Lincoln.

In the right-angled triangle ABC (with right angle at C) draw CD perpendicular to AB; also draw CE, CF, BM and AN bisectors of the angles ACD, BCD, ABC and CAB respectively.



We have then the following theorems:

Theorem I. *The bisector of the angle B is perpendicular to CE, and the bisector of the angle A is perpendicular to CF. Moreover, CE is bisected by the bisector of the angle B (in point a) and CF is bisected by the bisector of the angle A (in pt. b), and consequently ab is parallel to AB and is equal to* $\frac{1}{2}EF$.

Proof. Since

$$\angle bOd = \frac{A}{2} + \frac{B}{2} = 45°,$$

$$\angle bdO = \frac{BCD}{2} + \frac{B}{2} = 45°,$$

hence $\angle dbO = 90°$, and AN is perpendicular to CF. Similarly, it may be proved that BM is perpendicular to CE.

From the equality of the right-angled triangles FbA and AbC, we have $Fb = bC$, i. e., CF is bisected by the bisector of the angle A. Similarly, $Ea = aC$, and it follows at once that ab is parallel to AB and is equal to $\frac{1}{2}EF$.

Theorem II. *The bisectors CF and CE cut off on the hypotenuse AB segments AF and BE which are equal to the sides AC and BC respectively.*

Proof. From the equality of the right triangles FbA and AbC, it follows that

$AF = AC$, and from the equality of the right triangles BaE and BaC it follows that $BE = BC$.

THEOREM III. *The radius r of the circle inscribed in the triangle ABC is equal to* $\frac{1}{2}EF = ab$.

Proof. Since $CO = OF$ (ΔCOF is isosceles) and $CO = OE$ (ΔCOE is isosceles), $OE = OF$; and consequently EFO is an isosceles triangle.

Moreover, $\angle FOE = 2\angle FCE = 90°$. Whence, $OP \equiv r = \frac{1}{2}EF = ab$.

THEOREM IV. *The radius,* $r = cd/\sqrt{2}$, *i. e., the radius is equal to the side of a square whose diagonal is cd.*

Proof. From the similarity of the triangles aOb and dOc, we have

$$\frac{ab}{cd} = \frac{Oa}{Oc} = \frac{Ob}{Od} = \sin 45° = \frac{1}{\sqrt{2}}.$$

Hence (Theorem III), $r = ab = \dfrac{cd}{\sqrt{2}}$.

THEOREM V. *The line of centers of the circles inscribed in the triangles ADC and BDC is perpendicular to the bisector of the right angle C, i. e.,* $cd \perp CG$.

Proof. Since in the triangle cdC two altitudes cb and da pass through the point O, $CG \perp cd$.

THEOREM VI. *The centers of the circles inscribed in the triangles ADC and DBC, i. e., the points c and d are equidistant from the point P and their distances are equal to* $OP = r$.

Proof. From the equality of triangles OFd and OcE it follows that $OD = Ec$, and hence follows the equality of OPd and PEc and consequently $cP = Pd$. Moreover, $\angle OdF = 180° - (\alpha + \beta) = 180° - 45° = 135°$. Hence, if a circle be drawn with P as the center and $OP = PF = r$ as the radius $\angle OdF$ will be an inscribed angle and $Pd = cP = r$.

UNDERGRADUATE MATHEMATICS CLUBS.

EDITED BY R. C. ARCHIBALD, Brown University, Providence, R. I.

ADDRESS.[1]

By HERBERT E. HAWKES, Columbia University.

In the remarks which I have to make this morning, I do not propose to go into details regarding the history of Mathematics Clubs or to discuss the function of such clubs under ideal conditions. I wish rather to tell briefly of my own experience with such clubs which may possibly serve as a practical help to those who have conducted or are about to organize such organizations in their own institutions.

[1] Delivered at the second summer meeting of the Mathematical Association of America, at Cleveland, Ohio, September 6, 1917.

At the very start let me say that the kind of club I have in mind is not the kind which is of interest merely to the two or three ablest men in college. The exceptional man who even in his undergraduate days shows promise of unusual scientific enthusiasm and ability, the kind of man who appears once in several years in the small college, and of whom there is only one in a class in the large college, must be taken in charge by his instructor in any case, and should have careful personal direction so that if he is the kind of man who should be advised to study mathematics professionally, he may be brought in contact with the real thing in mathematical investigation as early as practicable. Such a man ought to be a leader in the club, but if the club is geared for this man, it will be a flat failure. The pace will certainly be too hot for his less gifted fellows, and the esult will be a mutual admiration society of about three somewhat unworldly spirits.

The kind of club I *am* talking about is one in which the able freshman can find some interest, and every student who is interested in interesting things and who has taken up the study of analytics can take an active part.

What is the use and the function of such a club as I have indicated? It seems to me that there are three kinds of purpose that such a club may serve, if it is successful. It surely benefits those students who present papers or solutions, it ought to help those who attend, and it should be a good thing for the college as a whole.

Just a word on this last point, before taking up the first two. One of the most unfortunate traditions among the undergraduates of our colleges is the feeling that they are serving their college only when they are playing baseball, or running the football team. Their loyalty is splendid and genuine, and such a precious enthusiasm should not be allowed to cool, even if it is sometimes observed to run in a direction of secondary importance. It is almost impossible for the undergraduate to realize that in the long run the greatest honor to a college is a competent and well-trained graduate, and the most far-sighted loyalty to his alma mater would lead him to prepare himself to be such a graduate. To be an officer of the mathematics club is an honor, and those who work for the welfare of the club are recognized as working for the college. And since its activities have to do with the intellectual life, it should appeal to the most genuine spirit of loyalty to the institution.

Let me emphasize, however, that unless the activities of the club are kept simple, so that it may have the reputation in college of being well within the domain of rationality of a good many students, this feature of the club will be a failure. For a small body of pale people talking about existence theorems is not a spectacle over which college students as a whole will enthuse. I am certain, however, that this idea of organizing certain extra-curricula activities which have contact with things of the mind helps to divert some of the energy of the students from an overwhelming development of athletics. I do not intend to express any lack of appreciation for athletics. Young men must have them. But few will argue that they are not carried to an extreme in many of our colleges, and that a counter-irritant should be sought.

The benefit of the club to the student who attends, but who has no stated part is fairly obvious. It depends, however, on the way the meetings are conducted, whether the auditor gets much out of it or not. In my experience it has been absolutely necessary to conduct the meetings in the most informal manner possible. The speaker expects to be interrupted, and to explain in greater detail questions of notation or demonstration or implication which anyone requests. Often the more carefully and clearly a paper is presented, the greater curiosity it awakens, and the oftener the speaker is asked to make more detailed explanation.

The fundamental fact is that students cannot be expected to go many times to a mathematics club because they feel that they ought to. They will only go because they want to, and nothing but a good time intellectually and socially will hold them. I do not mean by this that there should be meager food for thought, and much entertainment. Quite the contrary. The intellectual nourishment cannot be too good or too well prepared. But it must be adapted for their powers of digestion, and must be served with good fellowship and simplicity.

The benefit of the club to those who give papers is naturally greater than to any one else, and on them depends the success of the club to a large extent. Our responsibility as teachers of college mathematics is nearly as great in the direction of training our students in the difficult art of self expression as in the technique of mathematics. The regular class room instruction, especially in elementary work, consists to a large extent, for many of us, in patiently requiring our students to express the truth that is in them in the form of an English sentence which has a subject, a predicate and the proper qualifying clauses. Now the club affords a splendid opportunity to give a few good students, men who are worth taking pains with, excellent practice and training in the exposition of a specific topic in mathematics. I would remark parenthetically that this means an appreciable amount of painstaking and tactful work on the part of some one on the teaching staff. The students must feel that they do the whole thing, but they must be far from correct. I have rarely found the student who had really adequate terminal facilities. They always prepare papers from twice to four times as long as they should be, and find the greatest difficulty in condensing to advantage. Even if the student sees his problem in the proper perspective, it requires almost superhuman efforts to get him to express it clearly and concisely. But I am forced to say that assisting a student to get an effective method of exposition without doing the work for him is, in my opinion, scientific education of the most important and useful kind, and any means that enables one to render such service is an important feature of the college.

The second great benefit to the speaker in the mathematics club is the experience which he gets in examining the literature of his topic, and going outside his text books for his material.

The third, and the most important of all is the consciousness that comes with independent search for truth. The truth sought is usually not new to

science, but it is new to the student, and has all of the freshness of spring to him. For college students, what might be called "searchwork," that is, topics dealing with *important* questions which contain enough meat to interest any right thinking person, is much more profitable than the discussion of some topic which he has worked out for himself but which one cannot find in the books, or memoirs. In fact, I think that it is a pity for an able student to get the impression that the discovery of some little point that no one else has happened to mention, is mathematical scholarship and a substitute for a broad and wide knowledge of the subject.

My procedure with a student who has signified his desire to prepare something for the club is somewhat as follows. After finding a topic which is within his range, and in which he is likely to be interested, I give him one or two references which he is to read, and to report upon. Then I often tell him of other books in which he had better look, and urge him to go over all the literature he can find which seems to promise well. By the time this is done he can make an outline of what he proposes to say, specifying what he wishes to prove, and what is to be given without proof. The student always wants to prove almost everything, and cannot understand how a paper can omit proofs, and still be a good one, except when he is listening to it. Finally a few days before the meeting I try to go over with him in more detail the plan of his paper, and often the details of exposition. Of course he never speaks from notes, but puts diagrams on the board before the meeting. I have found that if a paper is planned which the student thinks will take twenty-five minutes, it actually takes forty-five if he is not interrupted badly. If he is interrupted, he never gets through, but has to summarize the last part. I make great efforts, however, to keep the papers down to twenty-five minutes of talk.

Undoubtedly the kind of organization which would be best adapted to students of one institution might not be suited to another. I have had experience with only two types of club at Columbia College. At first there was no organization at all. Meetings were held every two weeks by those who were interested to come, but without undergraduate officers, no dues, and no machinery at all. I always presided, in the most informal manner, and it was a purely spontaneous expression of interest in mathematics on the part of those who attended. After a couple of years of this sort of procedure, the students suggested that they have a regular club with officers, dues of twenty-five cents per year to cover the expense of sending cards to members, and a written constitution.

So far as the essential features of the club are concerned, I cannot see that it makes the slightest difference what kind of organization is in force. The fact from which one cannot get away, is that a mathematics club will not run itself, and that it requires a great deal of careful work to get speakers who can attract a good house. If the organization is such that some member of the staff can really look after this feature of the work and at the same time keep in the background so that the students seem to be in charge, it does not make much difference whether there are many officers or few, or whether there is a constitution or not.

I am inclined to think, however, that the club is of more service to the college community if there are undergraduate officers, and definite organization. If the professors can attend not as professors but as fellow creatures of the students there is no objection. But the moment that the boys feel that another disguised recitation is in progress, and that their professors are watching their accomplishment, the club is no longer a club, but a parade ground.

The most important and at the same time the most difficult feature of the club to manage, is the program. Especially during the first part of the year, papers must be provided which will attract the students and hold their interest. I suppose that the mathematical topics which naturally attract the student who is not far advanced, are those which he thinks have some mystic significance. Something about higher dimensions, or infinity, or paradoxes. Personally I think that the greatest service which a club can perform in regard to these topics is to denude them of their mystic qualities, and to bring the ideas involved squarely and fairly into relations with sound mathematics. But students cannot perform this service, for they have not the perspective, and so far as my experience goes, make a bad matter of any topic relating to these subjects. Consequently, it usually seems wisest for an older person to talk about the fourth dimension and orders of infinity, rather than a student.

One might think papers of an historical nature admirably adapted for presentation at the club. But to my great surprise, I have had very little luck with such papers. It may be my own fault, but I have found that the students do not have a sufficiently broad and scholarly grasp of the subjects whose history they are presenting to make topics of this kind rewarding. Here again, a member of the staff does much better.

A distinguished educator once complained to me that the teachers of mathematics and physics missed a great opportunity in not humanizing the presentation of their subjects. He said that a boy might study physics two years and not know whether Boyle was an Irishman or a very painful swelling on one's person. I have made a serious effort to get students to prepare papers on the "heroes of mathematics." They like the idea immensely but when it comes to preparing a paper showing just where Napier, Newton, Archimedes or Descartes stood in relation to what preceded and what followed, and what specific problem stimulated them to carry forward their epoch making work, I have sometimes been disappointed, owing again to the lack of perspective. But on the whole such papers have been much more successful than those which attempt to treat the development of an idea or set of ideas.

At least one excellent meeting a year can be provided by asking the students to bring paradoxes or puzzles of a mathematical nature. These may be presented, and solutions requested from the audience.

I have found no difficulty in finding enough interesting topics which involve little or no calculus. Such a procedure does not seem to result in the loss of interest on the part of the more mature students, for they feel the need of a more complete perspective over their elementary mathematics, and as a matter of

fact, a paper which involves a good deal of the calculus cannot be followed in detail by as arge body of students as one likes to have at the club. And, unlike the members of more pretentious mathematical societies, students do not enjoy sitting through a long paper, no part of which they understand with the possible exception of the title. And *they* will not do it.

I have mentioned several types of topics which students find some difficulty in presenting effectively. It is much easier to do that than it is to generalize regarding the kind of paper which is successful. For here, as in most questions involving education, it all comes back to the man behind the guns. If the student himself is thoroughly interested in a topic he can do wonders with it. If not, he can spoil the best subject in the world.

On the whole the students do best on topics which are very definite, and in which there is a clear relation between the new idea which they are presenting and some thing with which they are familiar. For example, I recall a very excellent paper on the exponential and logarithmic function of complex numbers, in which the well-known ideas of many-valuedness, and periodicity as found in trigonometric functions were taken for points of departure, and these same features of the functions under discussion were exposed. Also the paradoxes which arise when one takes the wrong branch of a many-valued function affords instruction and amusement.

Imaginary lines in the plane are also interesting. Any student of analytics can prove that the distance between two points on such lines is zero, and a number of other curious features. Here the hope of success of the paper lies in connecting the new ideas with the similar ones regarding real lines.

Another kind of topic which is likely to be successful is the graphical study of processes which are more or less familiar analytically. Such subjects as the graphical solution of equation, the study of constructions which are possible with the use of the compasses and a fixed parabola, construction with a two-sided straight edge are all excellent.

Since these remarks are simply intended to start a discussion in which it is hoped that many will take part, I do not propose to follow the example of the students in talking twice as long as I should. I must, however, express my conviction that a mathematics club in a college is a powerful stimulus on the side of sound scholarship, and to my certain knowledge able men are inducted through it into the serious study of mathematics, and stimulated in their ambition not only to know the subject so far as their abilities permit but to add to it as well. I know of no activity in which a college professor of mathematics can more profitably spend some effort, than in directing a club. For not only is he helping the cause of scholarship, but he is making a subject which has the reputation of being somewhat austere, an object of living and spontaneous interest on the part of a good body of students, and giving it a place in the college community.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF ALBION COLLEGE, Albion, Mich.

This club was founded in January, 1911, "to create a greater interest in present-day mathematics and allied subjects, and to study best methods of teaching mathematics." Any student showing unusual ability during the first three semesters of the college course may be elected to membership in the club. All nominations are made by the head of the department, Professor Edwin R. Sleight.

Meetings last for about an hour and the usual routine is as follows: (1) roll-call, each member responding to his name by a short discussion (2 or 3 minutes as a maximum)[1] of an assigned topic [there is never any trouble with this part of the program]; (2) short talk of the evening, limited to 10 minutes; (3) topic of the evening, occupying 20 to 30 minutes; (4) general discussion of any of the topics of the evening; (5) critic's report (the critic is usually a student).

Officers are elected at the close of each semester for the one following; those elected for the first semester of 1917–18 were: President, Lucille Ball '18; vice-president, Esther Turnell '18; secretary-treasurer, Myrtle Speese '19; program committee: Professor Sleight, the vice-president, the secretary-treasurer, and Gertrude Landon '19. For the second semester the officers were: President, Alice Money '18; vice-president, Myrtle Speese '19; secretary-treasurer, Floyd Harper '20.

Programs for the past two years are indicated below [(1) roll call, (2) short talk, (3) evening topic]. The short talks for 1916–17 were historical. "The programs for a normal year are so arranged that each member of the club gives one short talk and one long talk each semester. The programs for 1917–18 are not a fair sample of the work done by the club. We are so rushed by having classes six days of the week, and having practically no vacations that we decided that the programs for the year should, for the most part, be merely reviews of articles found in periodicals."

October 3, 1916: (1) None; (2) "Euclid" by Jesse Campbell '18; (3) President's address "The place of mathematics in the world" by Hazel Miller '17.

October 10: (1) "A recent invention to which mathematics was applied;" (2) "Archimedes" by Gertrude Landon '18; (3) "The forward movement in mathematics" by Alice Money '18.

October 17: (1) "Integration" [each person as his name was called was sent to the board and asked to integrate a form]; (2) "Apollonius of Perga" by Myrtle Speese '19; (3) "Some unusual integrals and their solutions" by Ralph Huffer '18.

October 24: (1) "A practical application of a geometrical theorem;" (2) "Diophantus of Alexandria" by Hazel Miller '17; (3) "A comparison of geometry with mechanics" by Gladys Harger '17.

[1] There were 19 members in 1917–18 and this number necessarily limited the time which each individual might occupy. Of the 19 members only 4 are young men. Before the war membership was about equally divided between men and women.

October 31: (1) None; (2) "Vieta" by Esther Turnell '18; (3) "Fourier's series and its application" by Professor Arthur F. Beal.

November 7: (1) "My mental attitude toward mathematics;" (2) "Descartes" by Alice Money '18; "The development of mathematics in the United States during the nineteenth century" by Myrtle Speese '19.

November 14: "Some interesting celestial objects" (illustrated lecture, open session) by Professor Beal.

November 28: (1) "A peculiar geometrical construction;" (2) "Gauss" by Vera Junkin '19; (3) "A geometrical representation of multiplication and division" by Mildred Chappell '19.

December 7: (1) "Mathematics in poetry" (quotations referring to mathematics); (2) "Newton" by Ralph Huffer '18; (3) "Mathematics and psychology" by Esther Turnell '18.

December 14: (1) "A coördinate system;" (2) "Leibnitz" by Gladys Harger '18; (3) "Homogeneous coördinate systems" by Gertrude Landon '19.

January 16, 1917: This program consisted of four short talks: "Right and wrong definition of a limit" by Lloyd Crippen '19; "Distributive law of multiplication geometrically proved" by Esther Turnell '18; "The teaching of factoring" by Gladys Harger '17; "Judging a teacher of mathematics" by Myrtle Speese '19.

January 23: (1) "Who's who in mathematics in America;" (2) "A simple solution of a Diophantine equation" by Lucille Ball '18; (3) "The circles of Apollonius" by Ralph Huffer '18.

February 6: (2) "Lagrange" by Myrtle Speese '19; (3) "Tests and short cuts in mathematics" by Jesse Campbell '18; (1) Application of the short cuts given in (3).

February 20: (1) "An interesting event in the life of some mathematician;" (2) "Euler" by Gladys Harger '17; (3) "Some simple applications of congruences" by Hazel Miller '17.

February 27: (1) "Mathematical current events;" (2) "Cauchy" by Edna Colwell '19; (3) "War and mathematics" by Vera Junkin '19.

March 6: (1) "Name of some subject in higher mathematics and its application;" (2) "Steiner" by Margaret Courtright '19; (3) "Different kinds of geometries, with special emphasis on descriptive" by Ian Patterson '19.

March 13: (1) "Mathematics applied to some definite problem;" (2) "Abel" by Mildred Chappell '19; (3) "The future of mathematics" by Mary Hutchins '19.

March 20: (1) "Some interesting personal experience in a mathematics classroom;" (2) "Hypatia" by Murray Fox '18; (3) "Relation of the teacher to the community" by Alice Money '18.

April 3: (1) "Some theorem used in algebra—statement and application;" (2) "Galois" by Gertrude Landon '19; (3) "Discussion of Fermat's theorem" by Ralph Huffer '18.

April 10: (1) "A brief review of some new text-book;" (3) Report by Edna

Colwell '19 who had been elected to attend the mathematics section of the State Schoolmasters' Club held at Ann Arbor.

April 17: (1) "Name of some mathematician connected with function theory, and a brief statement of what he did;" (3) "Teaching of mathematics in secondary schools" by Principal Harry R. Atkinson of Battle Creek, Michigan.

May 1: (1) "The name of some educational society, together with a brief statement of its purpose;" (2) "Sylvester" by Jesse Campbell '18; (3) "Variables and limits" by Lloyd Crippen '19.

May 8: (1) "Mathematical fallacies;" (2) "Weierstrass" by Ian Patterson '19; (3) "How to make the teaching of mathematics interesting" by Hazel Miller '17.

May 15: (1) "Some unusual function;" (2) "Lie" by Murray Fox '19; (3) "Eulerian integrals and gamma functions" by Lucille Ball '18.

May 22: (1) None; (2) "Origin of certain typical problems" by Esther Turnell '18; (3) "Poincaré and his contributions to mathematics" by Margaret Courtright '19.

May 29: Social evening and election of officers for the following semester.

October 16, 1917: President's address by Lucille Ball '18.

October 23: (1) "Mathematical current events;" (3) "Geometrical proof of formulas for sine, cosine and tangent of half angles" by Vera Junkin '19; "Algebraical developments in Ancient Greece and Babylonia" by Margaret Courtright.

October 30: (1) "Some theorem in college algebra proved;" (3) "Condition that three lines pass through the same point" by Joyce Hadaway '20; "Freshmen and freshmen algebra" by Esther Pearl '20.

November 6: (1) "Historical theorem known by the name of the discoverer;" (3) "Methods for solving irrational equations" by Esther Turnell '18; "The students viewpoint of calculus" by Alice Money '18.

November 13: (1) "A peculiar logarithmic combination;" (3) "Mathematics of warfare" by Ralph Huffer '18.

November 27: (1) "A peculiar geometrical construction;" (3) "A new proof for the law of sines" by Carla Kennedy '20; "Mathematics as a means of culture and discipline" by Elizabeth Hubert '20.

December 11: (1) "A mathematical fallacy;" (3) "Significance of mathematics" by Lucille Ball '18; "A historical account of mathematical induction"[1] by Myrtle Speese '19.

January 8, 1918: (1) "A recent text book;" (3) "Projective geometry—a historical account and some applications" by Mildred Chappell '19.

January 15: Election of officers.

February 26: (1) "The history and meaning of mathematical symbols;" (3) President's address: "Why mathematics should be studied in the high school" by Alice Money '18.

[1] Cf. F. Cajori, (1) *Bulletin of the American Mathematical Society*, Vol. 15, pp. 407–408, 1909; (2) "Origin of the Name 'Mathematical Induction,'" in this MONTHLY, Vol. 25, pp. 197–201, 1918.

March 5: (1) "A statement of a problem whose solution is the irrational root of a cubic equation;" (2) "Some suggestions on the teaching of geometry" by Dorothy Tichenor '20; (3) "Solutions of the cubic" by Almira Priest '20.

March 19: (1) "A mathematical puzzle;" (2) "Valid aims and purposes for the study of mathematics in secondary schools" by Don Alexander '20; (3) "The function of mathematics in scientific research" by Carlton Sawyer '20.

April 2: (1) "An original mathematical limerick;" (2) "The Perry method" by Joyce Hadaway '20; (3) "A review of some old arithmetics" by Carla Kennedy '20.

April 16: (1) "Helpful hints to the teacher;" (2) "The heuristic method" by Vera Junkin '19; (3) "The planimeter" by Gertrude Landon '19.

April 30: Social evening, and election of officers for next semester.

THE MATHEMATICS CLUB OF GOUCHER COLLEGE, Baltimore, Maryland.

This club of young women was founded in November, 1913, in order "to promote a spirit of comradeship in the department, to stimulate interest in mathematics and to provide an opportunity to discuss many topics which are not included in the regular course." All students who have completed the first courses in analytic geometry and in calculus, given the first semester of the sophomore year, are eligible for membership. During 1917–18 the club had 22 members and the average attendance at the meetings was about 15. There are no officers but Professor Florence P. Lewis usually presides and acts as program committee. "At the close of each meeting we have very informal discussion and very light refreshments furnished usually by the speakers of the evening."

"Previous to 1916–17 we had no course on the history and teaching of mathematics and so devoted the club work to these subjects. We tried to cover the history in a very elementary general way in two years of club work. A secretary was appointed to make of the work a typewritten résumé, copies of which were given to each girl."

November 12, 1917: "The parallel axiom and modern work on foundations of geometry" by Professor Lewis.

November 26: "Trisection of an angle and the duplication of a cube" by A. Marie Whelan '18.

December 10: "A demonstration lesson in the teaching of geometry" by Miss Elizabeth White, teacher of mathematics, Eastern High School, Baltimore, Md.

January 21, 1918: "The triangle and its circles"[1] by Margaret Amig '19.

February 18: "Geometry of four dimensions" by Marguerite Lehr '19.

March 4: "The history of the invention of logarithms" by Teresa Cohen '12, graduate student of Johns Hopkins University.

[1] See this MONTHLY, 1918, p. 231, note 2.

April 15: "Construction of a pentagon" by Anna L. Ellery '20; "Fallacious proof that all triangles are equal" by Ethel H. Fox '19; "Orthogonal circles" by Effie M. Gray '18.

April 29: "Illustration of non-Euclidean geometry on the Poincaré sphere" by Professor Clara L. Bacon.

May 18: Club picnic at Herring Run.

May 20: "Graphic methods of presenting facts" by Mildred Grafflin '20; "A new theorem on equal circles"[1] by Ethel R. Carroll '20; "History of famous problems in algebra" by Ida R. Marshall '20.

TOPICS FOR CLUB PROGRAMS.

13. Constructions with a Double-Edged Ruler.

It is just about one hundred years since the French mathematician and engineer Jean Victor Poncelet wrote his famous *Traite des propriétés projectives des figures*[2] which entitles him to rank as one of the greatest contributors to the development of modern geometry. On pages 187–190 are given indications of the proofs that: *Every geometrical construction with ruler and straight edge is also possible*: (1) *with straight edge alone if a single circle and its center have been drawn in the plane* (a theorem often incorrectly attributed to Steiner as originator[3]); and (2) *with an angle*[4] (of wood say) *of given opening* α, *alone.*

Particular cases of the double-edged ruler (2) are: (3) a square, when $\alpha = 90°$; and (4) a ruler with parallel edges when $\alpha = 0°$.

As a proof of (1) Steiner found it convenient to show that the means at his disposal were sufficient for solving the following problems:[5] (*a*) To draw through a given point a parallel to a given line; (*b*) to produce a given line segment its own length any given number of times or to divide it into any number of equal parts; (*c*) to draw through a given point a line perpendicular to a given line; (*d*) to draw through a given point a line which makes with a given line a given angle; (*e*) to bisect a given angle or to make an angle any number of times greater; (*f*) to draw from a given point in a given direction a segment equal to a given segment; (*g*) to determine the points of intersection of two circles whose radii and centers are given; and (*A*) to determine the points of intersection of a given line with a circle the position of whose center and the length of whose radius alone are given.

Recent proofs of (2), (3) and (4) have been, in effect, reduced to showing that these same constructions could be carried through by the special instruments in

[1] See this Monthly, 1918, p. 231, note 3.

[2] Paris, 1822; see also 2e éd., tome 1, 1865, pp. 181–184, 413–414. Poncelet tells us that this great work was the result of research made in 1813–14 when in Russian prisons "privé de toute espèce de livres et de secours, sur-tout distrait par les malheurs de ma patrie et les mien propres."

[3] Hence the expression, "Steiner constructions" employed in this connection should be "Poncelet constructions."

[4] "Fausse équerre."

[5] J. Steiner, *Die geometrischen Constructionen ausgeführt mittelst der geraden Linien und eines festen Kreises*, Leipzig, 1833, pp. 2–3, etc.

question. Such is the mode of procedure of Adler in 1890 and 1906,[1] of Enriques in 1903,[2] of Giacomini in 1907[3] and of Killing and Hovestadt in 1910.[4]

For separate discussion of constructions by means of a ruler with parallel edges reference may be given to papers and notes by Lebescond de Coatpont and De Tilly,[5] Marenghi, and Concina.[6]

Since a parallelogram may be drawn at once with a double-edged ruler note that the following famous problem was discussed by Lambert,[7] s'Gravesende,[8] Poncelet,[9] Tractenberg[10] and Child[11]: "Given a parallellogram, construct with straight edge alone a parallel to a given line."

Important practical applications, in surveying and warfare, of constructions involving a double-edged ruler are indicated in the remarkable little book of F. J. Servois, *Solutions peu connues de différens problèmes de géométrie-pratique* . . . A. Metz, An XII [1804], (especially pages 68–79), and in the very interesting work of Gohierre de Longchamps to which reference has been made already. Solutions of surveyors' problems with the square were given also by L. Mascheroni in his *Problemi per gli agrimensori con varie soluzioni*, In Pavia, MDCCXCIII.[12]

With ruler and compasses, only special problems of degree higher than the second can be solved, no difference how many of these instruments may be em-

[1] A. Adler, (1) "Über die zur Ausführung geometrischer Constructionsaufgaben zweiten Grades notwendigen Hilfsmittel," *Sitzungsberichte der mathematisch-naturwissenschaftlichen Classe der kaiserlichen Akademie der Wissenschaften zu Wien*, Band 99, Abtheilung IIa, Jahrgang 1890, Heft 8; (2) *Theorie der geometrischen Konstruktionen*, Leipzig, 1906, pp. 123–138.

[2] F. Enriques, *Vorlesungen über projektive Geometrie*, Leipzig, 1903, pp. 266–277; 2. Aufl. Leipzig, 1915, p. 254.

[3] A. Giacomini, "Über die Lösung der geometrischen Aufgaben mit dem Lineal und den lineal Instrumenten . . .," pp. 95–103 of *Fragen der Elementargeometrie* gesammelt und zusammengestellt von F. Enriques, II. Teil, Leipzig, 1907; Italian edition, Bologna, 1914, pp. 89–96.

[4] W. Killing und H. Hovestadt, *Handbuch des mathematischen Unterrichts*, Band I, Leipzig, 1910, pp. 194–199.

[5] "Sur la géométrie de la règle," *Nouvelle correspondence mathématique*, tome 3, 1877, pp. 204–208; tome 5, 1879, pp. 439–442; tome 6, 1880, pp. 34–35.

[6] C. Marenghi, "Geometria della riga a due orli paralleli," *Il bolletino di matematiche e di scienze fisiche e naturali*, Bologna, Vol. 2, 1900–1901, pp. 129–145; U. Concina "Resoluzione dei problemi fondamentali relativi al trasporto delle figure piane colla riga a due orli paralleli," *idem*, 1901, pp. 225–237.

[7] J. H. Lambert, *Freie Perspective*, 2e éd., Zürich, Vol. 2, p. 169. This solution is given also in: (1) *Mathematical Questions and Solutions from the 'Educational Times,'* Vol. 57, London, 1892, p. 88; and (2) J. W. Russell, *An Elementary Treatise on Pure Geometry*, New and revised edition, Oxford, 1905, p. 318.

[8] W. J. s'Gravesende [died 1742], *Oeuvres philosophiques*, Amsterdam, 1774, Ire partie, § 312, p. 174.

[9] *Traité des propriétés*, etc., 1822, pp. 106–107; Lambert's solution is also given here. Both of these solutions are reproduced in L. Cremona, *Elements of Projective Geometry*, translated by C. Leudesdorf, Oxford, 1885, pp. 96–97; Poncelet gave his solution of the *Traité* and two others in his *Applications d'analyse et de géométrie*, Paris, 1862, pp. 437–439 (this part of the work was written while he was in prison in 1813–14). One of these solutions is also reproduced in G. De Longchamps, *Essai sur la géométrie de la règle et de l'équerre*, Paris, 1890, pp. 232–235.

[10] M. I. Trachtenberg, *Mathematical Gazette*, 1908, Vol. 4, p. 334.

[11] J. M. Child, *Mathematical Gazette*, 1910, Vol. 5, pp. 283–284.

[12] Second edition with title: *Problemi di geometria colle dimostrazioni del Capitaus Sacchi*, Milano, 1803; third edition, 1832. French translation of the first edition: *Problèmes pour les arpenteurs avec différentes solutions*, Paris, An XI–1803; 2e éd., with title: *Problèmes de géométrie pratique pour les arpenteurs avec différentes solutions*, Paris, 1838.

ployed. In contrast to this it is interesting to note, as Adler (*l. c.*) emphasizes, that the right angle (square) is a much more powerful instrument. Indeed Plato's instrument for the solution of the problem of the duplication of a cube[1] is equivalent to the use of two right angles to solve a binomial cubic equation. So also four right angles could be used to solve a fifth degree binomial. Descartes employed a similar instrument with right angles[2] for the insertion of several geometric means between two line segments.

The fundamental ideas here introduced were adapted by Lill[3] so as to extend to the solution of polynomial equations with numerical coefficients. From this Adler readily formulated, in particular, the result: Problems of the third and fourth degree can be rigorously solved geometrically by means of several right angles.

COLLEGIATE MATHEMATICS FOR WAR SERVICE.

SEND WAR SERVICE COMMUNICATIONS TO DR. HENRY BLUMBERG, University of Illinois.

FIRING DATA.

By J. K. WHITTEMORE.

In the first part of this paper, I shall outline a course in "Firing Data," and in the second part, make some suggestions as to the conduct of such a course. The outline is based on the course given in the last college year in the Yale R. O. T. C. The course in firing data includes all the mathematics necessary for an officer of the Field Artillery, and is an extremely important part of the training for a commission in that service. The work, as here described, applies to the U. S. 3-inch gun, the British 18 pounder, and the French 75 mm., but the tables used in the examples apply only to the U. S. 3″ gun; they are given in Danford and Moretti's "Notes on Training Field Artillery Details," Yale University Press, Sixth Printing, Feb., 1918.

The fundamental firing data problem is the computation of data for the battery for the opening salvo in indirect laying from observations made at the battery commander's station. These data are range, site, deflection and deflection difference. Indirect laying is pointing a gun at a target invisible at the gun; both target and battery must be visible at the battery commander's station.

The course begins with certain definitions which must be thoroughly learned

[1] M. Cantor, *Vorlesungen über Geschichte der Mathematik*, 3. Auflage, Band I, Leipzig, 1907, pp. 227, 353. *Cf. L'Algèbre d'Omar Alkhayyâmî* publiée . . . par F. Woepeke, Paris, MDCCLI, pp. 94–96; A. Conti, in *Fragen der Elementargeometrie* (Enriques), Band 2, p. 215; and A. Adler, *l. c.*, 1906, p. 237.

[2] *La Géometrie*, livre II, *Oeuvres de Descartes* publiées par C. Adam et P. Tannery, Vol. 6, Paris, 1902, p. 391.

[3] E. Lill, (1) Résolution graphique des équations numériques de tous les degrés à une seule inconnue, et description d'un instrument inventé dans ce but," *Nouvelles annales de mathématiques*, 1867, (2), tome 6, pp. 359–362; (2) "Résolution graphique des équations numériques d'un degré quelconque à une inconnue," *Comptes rendus de l'académie des sciences*, Paris, tome 65, pp. 854–857. See also T. Vahlen, *Konstruktionen und Approximationen in systematischer Darstellung*, Leipzig, 1911, p. 141.

by the student. These are given alphabetically in Drill Regulations for Field Artillery and in Danford and Moretti. This arrangement is convenient for reference, but the definitions should be grouped in proper connection for instruction. For the comprehension of this paper it is necessary to give the following:

Mil, a unit of angular measure, is one sixteen hundredth of a right angle, and is approximately one thousandth of a radian. The numerical measure of an angle in mils exceeds one thousand times its radian measure by about two per cent. The usefulness of the mil depends on the formula for small angles, $m = c/k$, where m is the angle in mils subtended by a line c units long at a distance of 1,000 k units. If c is the side of a right triangle opposite the small angle m mils, and 1,000 k the adjacent side, then m is given by the formula with an error not greater than two per cent. if m is not greater than 330. The formula is exact for a certain value of m nearly equal to 270. The same formula is used for small m when 1,000 k is not the leg but the hypothenuse of the right triangle. In this case the value of m given by the formula is always too small, the error varying from two per cent. at zero to about three per cent. for $m = 300$.

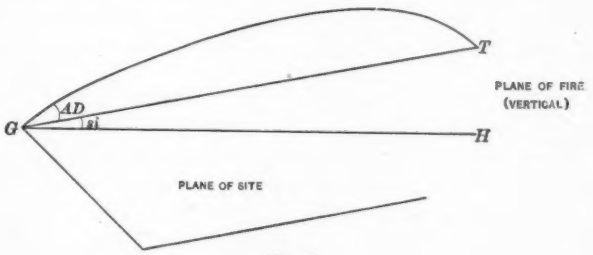


FIG. 1.

Plane of site is the plane containing the line GT and the horizontal line perpendicular to the axis of the bore of the directing gun. G is the muzzle of the directing gun for which the data are computed, and T, the target. (Fig. 1.)

Site (*si*) is the angle in mils of the plane of site with the horizontal plane. It is positive or negative according as T is above or below G. To avoid the use of negative numbers we use SI, where $SI = 300 + si$. The term site is used for other elevations; thus we speak of the site of the target at the battery commander's (BC) station. (Figs. 1 and 2.)

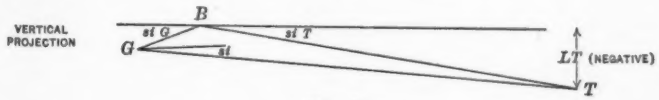


FIG. 2.

Angle of departure (AD) is the angle in mils of the tangent to the trajectory of the projectile at the point of departure from the gun with the plane of site. (Fig. 1.) For most service conditions in the Field Artillery the trajectory may

be regarded as rigid without important error, so that change of site produces no change in the relation of range and *AD*. Actually the elevation of the gun is given by two mechanisms, on one of which the site is laid off, and on the other, the range.

Aiming point (*P*) is the object on which the panoramic sight of *G* is directed. (Figs. 4 and 5.)

Deflection (*D*) is the angle *PGT* measured counterclockwise (or the angle *TGP* measured clockwise) as set off on the panoramic sight. (Figs. 4 and 5.)

Deflection Difference (*DD*) is the difference in deflection applied to other than the directing gun of the battery necessary to bring it to bear on the proper portion of the target. For parallel fire *DD* is the parallax of the aiming point. (Fig. 6.)

Distribution is the lateral relation of the shots of a salvo.

After the definitions necessary have been well learned, there should be considerable practice in the use of mils. One important problem depends on nothing else, two others only on the use of mils and the range table. The first of these is the determination of the site from observations at the B. C. station (*B*). For this the formula (Fig. 2)

$$SI = 300 - \frac{LG - LT}{R}$$

may be used, where *LG* and *LT* are the levels of *G* and *T* above *B*, and *R* is one thousandth of the range. These levels are negative if *G* and *T* are below *B*, and this fact often leads to confusion. I think it better to not use any formula for this problem, rather to use a figure and to describe relative elevations as above or below. Such a problem is usually formulated as follows: *SI G* at *B* = 260, *SI T* at *B* = 285, *BG* = 270, *BT* = 3000, *Rn* = 3100. Find *SI*. We have *LT* = − 45, *T* 45 below *B*, *LG* = − 11, *G* 11 below *B*, *T* 34 below *G*, $si = \frac{-34}{3.1} = -11$, *SI* = 289.

The next of these problems is that of the clearance of a crest of known height and distance when firing on a target whose site and range are given. The clearance is given in mils by the formula

$$AD + si - \frac{c}{k} - ADC,$$

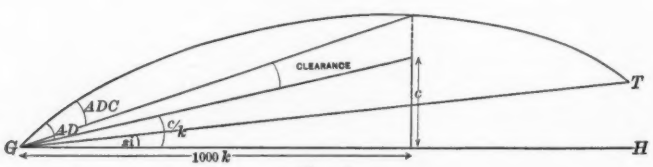


FIG. 3.

where *c* is the height of the crest in yards, 1,000 *k* the distance or range of the crest in yards, and *ADC* the angle of departure for the range of the crest. If

the value given by the formula is negative, the crest is not cleared. If we set the clearance equal to zero and solve for AD we have the angle of departure for the minimum range for the crest. The proof of this formula follows immediately from Fig. 3, provided that all angles involved are small enough to permit the use of the mil formula, as is usually the case in practice. An example of this kind requires evidently the use of a range table connecting AD and Rn (Danford and Moretti, page 254). Such a problem is usually formulated as follows:

Crest 20 yards high, 400 yards from battery; SI 295, Rn 2600. How much is crest cleared in mils and yards? What is minimum range for crest?

The clearance is $76 - 5 - 50 - 7 = 14$ mils, 5.6 yards.

AD for minimum range is 62, 2,250 yards.

The third problem is that of the proper location of the battery to the rear of a crest for a given target. It may be shown, if the tangent to the trajectory for the given target from a gun supposed on the crest is produced backwards to intersect the level ground to the rear of the crest, that a battery placed at the point of intersection will always clear the crest when firing on the given target. Such a problem is the following:

Crest is 15 yards above ground to the rear. Observer on crest finds for T, SI 290, Rn 3500. Where should guns be located? What is SI? How much is crest cleared?

Guns should be to rear of crest $1{,}000 \times \frac{15}{115 - 10} = 140$ yards. $SI = 300 - \frac{20}{3.64} = 294$. Clearance is $122 - 6 - 105 - 2 = 9$ mils, 1.3 yards.

These three kinds of problems are of considerable importance, furnish good practice in the use of mils, and give the student an opportunity to become familiar with the range table. I do not think that these problems should be once treated and practised and then laid aside, rather the student should be required to work at them occasionally throughout the whole course.

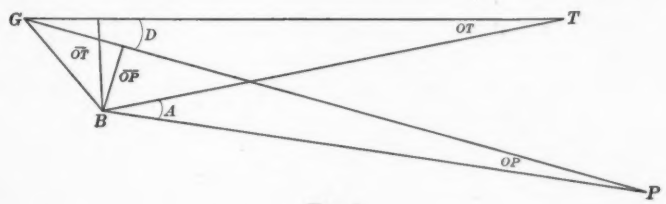


FIG. 4.

The most important of the problems of firing data is the calculation of the deflection for the directing gun, usually no. 2, as we count from the right when facing T. This problem should be introduced as early as possible and should be dwelt upon and hammered in until the students can do one in two minutes and until they consider it the easiest thing in the course. In Figs. 4 and 5, G is the directing gun, T the target, P the aiming point, B the B. C. station; the deflec-

tion D is the angle PGT measured counterclockwise, and is evidently greater or less than 3,200 mils according as P is to the left or right of GT; the angle PBT, which is called A, is measured at B; the angles at T and P, which in practice are small, are the *offsets* (offset angles) of target and aiming point respectively; the perpendiculars from B to GT and GP are also called offsets or offset distances. It is easily seen that

$$D = A \pm OT \pm OP,$$

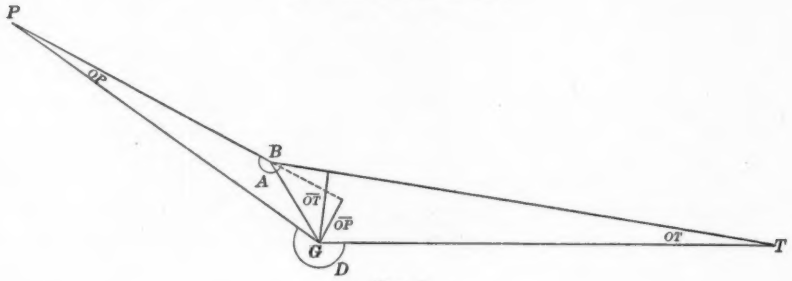


FIG. 5.

OT and OP denoting offset angles of target and aiming point respectively. The offset angles are found in mils from the offset distances,

$$OT = \frac{\overline{OT}}{\frac{1}{1000}BT} \qquad OP = \frac{\overline{OP}}{\frac{1}{1000}BP}.$$

It is clear that these formulas gain in accuracy by choosing B near G and P remote from B. In practice examples the distances BT, BP, $\overline{OT}$, $\overline{OP}$, are at first given. The range GT differs little from BT if B is near G, and may be taken as BT with a small correction, to be added or subtracted according as B is in front of or behind the line of guns, and given to the nearest hundred yards. The chief difficulty of the problem is in giving proper signs to the offset angles in the formula for D. These signs may be determined from a figure but this does not always eliminate the difficulty, as angles greater than a straight angle must frequently be used. I have used the following rules and believe they save time and make for accuracy:

1. B to left of GT, $+ OT$; right, $- OT$.
2. P behind BG, OP same sign as OT; front, opposite sign.

If in Fig. 4, we suppose $BT = 3{,}600$, $BP = 4{,}800$, $BG = 240$, $\overline{OT} = 200$, $\overline{OP} = 160$, $A = 710$, we have $OT = 56$, $OP = 33$, $D = A - OT + OP = 690$ (687), deflection being given to the nearest ten. $Rn = 3{,}700$.

In field practice, of which I shall say more later, the distances BT and BP are measured with a range finder, or may be taken from a map, or in case of necessity, estimated, *not guessed;* BG may be measured or estimated, or computed as follows: the gunner lays off 10 or 20 yards at G in a line perpendicular to BG, the angle subtended is measured at B, and BG is then computed from the mil

formula. But it is not easy to measure or to estimate accurately the offsets OT and OP. For this reason the method is modified (Fig. 5) by drawing the offset lines from G perpendicular to BT and BP; the offsets are then found from $\overline{OT} = BG \sin GBT$, $\overline{OP} = BG \sin PBG$; the range may if necessary be computed more accurately from $Rn = BT - BG \cos GBT$. This method requires the measurement of *two* angles at B. I think it is not desirable in this work to use the trigonometric functions of any but acute angles. The sign of the correction to BT for range can be seen at once, and is negative or positive according as B is to the rear or in front of the guns. The trigonometric work need not be accurate to more than one per cent. A trigonometric table of sines and cosines giving these functions to two decimals for every hundred mils is adequate, and may easily be written on a small card. It is most desirable and important that the student acquire great speed as well as accuracy in working deflection problems.

The deflection difference (DD) for parallel fire given to a battery with guns at equal normal intervals of 20 yards is that for gun no. 3 with no. 2 as directing gun. The DD for no. 4 is double that for no. 3, DD for no. 1 is that for no. 3 with the sign reversed. The DD given is positive or negative according as P is in front or to the rear of the line of guns, and is given to the nearest five mils. It is evident from Fig. 6, which is not drawn to scale, that in mils

$$DD = \frac{20 \cos D}{\frac{1}{1000} GP}.$$

In practice average values of cos D, called obliquity factors, are used, each running for 400 mils. These factors are 1, .9, .7, .4, 0; the factor 1 runs from $D = 6{,}200$ through 6,400 to 200. The obliquity factors may conveniently be incorporated in the card trigonometric table whose use is suggested. The

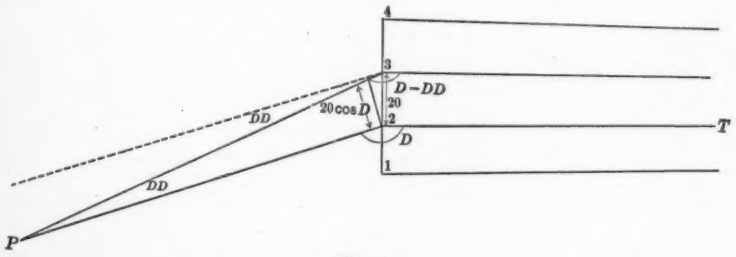


FIG. 6.

simplest method of computing DD is to write first the sign, second the obliquity factor, then $20/\frac{1}{1000} GP$, compute, and give result to the nearest five. In practice, since BP is known, and generally differs little from GP, BP may be used in place of GP. Suppose, for example, DD is required for parallel fire for battery at normal intervals; given GP 2,500 yards, D 2,300.

$$DD = -.7\frac{20}{2.5} = -5, (-5.6).$$

An important type of problem, requiring only the use of the mil formula and the idea of *DD*, is the problem of distribution. Such problems are rather in "fire control" than in firing data, having to do with the correction of data from observation of the results of firing, but they have practical value and are of use to the student, and may well be considered in connection with the work in *DD*. They appear to be extremely confusing to the student, and for this reason I give in detail the solution of such a problem explaining what seems to me a good method of arranging the work. The following problem, slightly modified, is given in Danford and Moretti, no. 19, page 265:

Target is a battery at twice the normal interval, *Rn* 4,000. Battery is in position with intervals

. | . 24 . | . 8 . | . 16 . | .

A. P. is 2,500 yards distant, *D* is 2,300, and *DD* for parallel fire, guns at normal intervals, has been given to the guns. At the first salvo, the right gun is on its target. Give commands necessary to put the other guns on their proper targets. Draw diagram.

The diagram I draw as follows: The line of guns is represented on a vertical line at the left of the paper, the line of fire to the right across the paper, the target a vertical line on the right, *T* the point of impact of no. 2, the directing gun, the range written on the line of fire. The next step is to compute *DD* as given. This was found in a preceding paragraph to be − 5. Determine next the point of impact of each gun with reference to *T*, points above *T* being represented by *T* +, below *T* by *T* −. We require for this the correct *DD* for parallel fire for each gun. These are respectively for nos. 3, 4, 1,

$$-7\frac{8}{2.5}=-2, \qquad -.7\frac{32}{2.5}=-9, \qquad +.7\frac{16}{2.5}=+4,$$

each computed to the nearest mil. These *DD* if used would give respectively as points of impact $T+8$, $T+32$, $T-16$. The *DD* actually used are respectively -5, -10, $+5$. The differences between the correct *DD* for parallel fire and those used produce changes in yards at 4,000 yards of -12, -4, $+4$, so that actual impact on the line of the target are for 1, 2, 3, 4 respectively

$$T-16+4=T-12, \quad T, \quad T+8-12=T-4, \quad T+32-4=T+28.$$

No. 1 is on its proper target; the proper targets of the other guns are the opposite guns of the enemy battery at 40 yard intervals, and are therefore $T+28$, $T+68$, $T+108$. Guns 2, 3, 4 should then be turned to the left, or their points of impact on the line of *T* raised, by 28, 72, and 80 yards respectively, that is their deflections, for 4,000 yards, increased by 7, 18, and 20 mils respectively. The actual work is conveniently arranged in lines, one for each gun:

No. 3 $DD=-.7\frac{8}{2.5}=-2$ Impact $T+8-12=T-4$ should be $T+68$ left 18

No. 4	$=-.7\frac{32}{2.5}=-9$	$T+32-4=T+28$	$T+108$	20
1	$=+.7\frac{16}{2.5}=+4$	$T-16+4=T-12$		
2		T	$T+28$	7

The last problem to be considered is also a problem of fire control, the correction of the firing data from the observed effect of a number of shots by an elementary application of the theory of probability. The theory of probability here used has no direct connection with the subject of probability as taught in "college algebra," but is based on the law of errors. Problems dealing with range, deflection, and height of burst are solved in the same way. I shall illustrate the method by solving a problem in the correction of range. The center of impact is a point in the line of fire such that an equal number of shots fall short of it and over it. A zone is an interval on the line of fire of which the center of impact is the center; the sixty per cent. zone, for example, is the zone within which sixty per cent. of the shots fall. The probability factor for any zone is the ratio of the width of that zone to the width of the fifty per cent. zone. A table of probability factors, the same for range, deflection, and height of burst, is given in Danford and Moretti, page 223. The probable error is one half the width of the fifty per cent. zone, and is the distance from the center of impact as often exceeded as not by the points of impact of all shots fired. The probable error varies with the range. A table of probable errors in range, deflection, and height of burst, for ranges from 1,500 to 5,000 yards, is given on page 220 of the same book.

Suppose it is observed in firing 12 rounds (shots) at *Rn* 2,000 that 10 fall short, 2 over the target. What change in range should be made?

We reason as follows: $\frac{1}{6} = 16\frac{2}{3}\%$ of the shots fall over; then T is at the further edge of the $66\frac{2}{3}\%$ zone. For *Rn* 2,000 the probable error in range is 34 yards; the probability factor for the $66\frac{2}{3}\%$ zone is 1.44. Then T is $1.44 \times 34 = 49$ yards beyond the center of impact. The range should be increased by 50 yards, no changes of range other than multiples of 25 yards being possible. Tables of probable errors and probability factors may be and should be dispensed with, for an officer of the Field Artillery should be as far as possible independent of tables. No considerable error will be made if the probable errors for all ranges in range, deflection, and height of burst are taken respectively as 30 yards, 2.4 mils, 1.8 mils. It may be a help to the memory to note that these figures are 5, 4, 3×6. These probable errors apply only to the U. S. 3″ gun. The probability factors for the 50%, 82%, 96% zones are respectively 1, 2, 3 nearly; these zones correspond to 25%, 25% + 16%, 25% + 16% + 7% respectively of shots beyond or short of the center. These figures are easily remembered if we note $2 + 5 = 1 + 6 = 7$. The probability factor required in any example is easily found with sufficient accuracy by interpolation. Thus if we work the example given above without consulting the printed tables

the probability factor is found for the 33⅓% of shots between T and the center of impact as $1 + 8\frac{1}{3}/16 = 1.52$. The probable error being taken as 30 yards, the change to be made in range is again 50 (46) yards. While this work may seem very rough to the mathematician he may feel assured that it is of the greatest practical value.

I wish in the following pages to make certain general remarks concerning the course in firing data, which will, I hope, be of interest to any one planning such a course.

The only mathematical knowledge required of a student taking the course is arithmetic and a very little of each of the following, algebra, plane geometry, plane trigonometry. To do well in the course the student must learn to do accurately and quickly such multiplications and divisions as have appeared in the examples worked in the preceding pages. It is desirable that he should have a sense of accurate approximation, that he should learn to carry his arithmetical work far enough and not waste time by carrying it further. Speed in arithmetical work, as in every part of the work in computing firing data, should be a constant aim in the instruction. In algebra the student must be able to substitute numbers in such simple formulas as have appeared in this paper, to transform simple equations, and to handle negative signs without blundering. No geometry seems necessary beyond an appreciation of the equality of alternate-interior angles. In trigonometry it is necessary only to know how to find the legs of a right triangle, given the hypothenuse and one acute angle.

The time devoted to the course last year at Yale was three hours a week for half the college year. It was felt that this time was not enough for the students, mostly freshmen, to become thoroughly familiar with the problems of the course, and was certainly not enough for the students to learn to work these problems with proper speed. It was planned in the coming year to give to freshmen in the R. O. T. C. a three-hour course throughout the college year on trigonometry and firing data. Probably more than half the time in such a course would be given to firing data.

A very important part of the course is field work. It was not possible last year to give the students nearly enough practice outdoors. My division had five afternoons in the field and should have had at least twice as much. For outdoor work, "fire control" instruments, a range finder, and an aiming circle or scissors are very desirable, but probably cannot be obtained at present. If these instruments are not available the distances from the B. C. station to target and aiming point should be given to the students by the instructor or obtained by the students from a map, or may be estimated, if proper methods of estimation are taught. Such methods are given in Danford and Moretti, pages 146–149. Angles may be measured with some accuracy by a B. C. ruler, which is six inches long divided into sixty equal parts; to the center is attached a string twenty inches long. Each division of the ruler subtends an angle of five mils when the ruler is twenty inches from the eye, the whole ruler thus subtending 300 mils. Angles greater than a right angle should not be measured with a B. C. ruler; if

such an angle is required, a straight line should be run and the supplementary angle measured. To measure an angle of site it is necessary to establish a horizontal line. A simple and fairly satisfactory way to do so is the following: Construct accurately on a sketching board a line perpendicular to one edge and attach a string to a point of this line; in using, attach a weight to the other end of the string and move the board in a vertical plane until the string hangs along the line; the edge is then horizontal. Some distant object in a horizontal line with the observer, as nearly as possible in the same vertical plane with the observer and the point whose site is to be measured, may thus be determined, and the site measured with the B. C. ruler with an error probably not greater than five mils. Angles may be roughly measured by sighting over the knuckles of the clenched fist held at arm's length. Of course each observer must for this purpose know the scale of his own fist.

The course in firing data should if possible be coördinated with other courses, the outdoor work particularly with map making and panoramic sketching and with use of fire control instruments if these are available. For any part of the work, instruction in map reading and matériel (the mechanism of the gun) is useful. The course should be followed by a course in fire control, but the latter can probably be given satisfactorily only by a trained artillery officer.

The tables needed for the course have already been mentioned. They are range table, tables of probable errors and probability factors, and table of sines and cosines for angles in mils. All are given by Danford and Moretti. A brief but sufficiently accurate range table for the U. S. 3″ gun is here given. It seems however undesirable to lay emphasis on memorizing either the range table or the table of probable errors, since these apply only to the U. S. 3″ gun. In the following table the first column is range in yards, the second is the corresponding *AD*, the third, the change in *AD* for 100 yards' increase in range.

1000	20
2000	50
3000	90
4000	140
5000	200
6000	270

				SIN	COS				
6400	3200	1	3200	0	0	1.00	1600	1	4800	4800
6300	3300	1.	3100	100	0.10	0.99	1500	0	4700	4900
6200	3400		3000	200	0.20	0.98	1400		4600	5000
6100	3500	.9	2900	300	0.29	0.96	1300	.4	4500	5100
6000	3600		2800	400	0.38	0.92	1200		4400	5200
5900	3700		2700	500	0.47	0.88	1100		4300	5300
5800	3800	.7	2600	600	0.56	0.83	1000	.7	4200	5400
5700	3900		2500	700	0.63	0.77	900		4100	5500
5600	4000		2400	800	0.71	0.71	800		4000	5600
				COS	SIN					

In the trigonometric table above the sines and cosines for every hundred mils are given to two decimals. The obliquity factors for corresponding deflections are also given.

Of course it is necessary that the students work a large number of numerical problems. Danford and Moretti give a considerable number (pp. 263–282) taken from examinations held in the non-commissioned officers' course at the school of fire. The instructor will have no difficulty in making up any number of similar problems.

While there exists a large amount of literature on the subject of firing data, I believe that for an introductory course nothing further is necessary than the Drill and Service Regulations for Field Artillery and some text such as that of Danford and Moretti.

Requests for further information concerning the course in training for service in the Field Artillery at Yale may be addressed to the Curriculum Committee, Artillery Hall, New Haven, Conn.

YALE UNIVERSITY.
September, 1918.

COURSES IN NAVIGATION AT THE UNITED STATES NAVAL ACADEMY.[1]

The Department of Navigation at the Naval Academy gives two different courses, the regular course for midshipmen, and a special course for Reserve Officers. The two courses are similar in content, but differ greatly in method of presentation and in the amount of time allotted.

The course for midshipmen is preceded by a thorough course in spherical trigonometry and stereographic projections given by the Department of Mathematics. To this preliminary work about thirty-five recitations are devoted, approximately half of this time being given to the use of stereographic projections and to the logarithmic solution of the same problems treated by projection. The text book is Brown's Trigonometry and Stereographic Projections, published by the Naval Institute, Annapolis, and the tables used are Bowditch's "Useful Tables," from the text on navigation by this author.

The work under the Department of Navigation involves three recitations per week during the last three terms (semesters) of the academic course. In the spring term preceding the first class (senior) year, two months are devoted to a brief course in astronomy, based on White's Theoretical and Descriptive Astronomy (John Wiley and Sons), and the remainder of the term is given to such work in navigation as will best prepare the student for the practical work of the summer cruise. During the last year the whole field of navigation and compass deviations is covered with Muir's Navigation and Compass Deviations (Naval Institute, $4.20) as a principal basis. Use is made of the Practical Manual of the Compass prepared at the Naval Academy (Naval Institute, $1.75), also of Bowditch's American Practical Navigator, the Nautical Almanac, and Azimuth Tables, all publications of the Hydrographic Office, Navy Department, Washington, which may be purchased through the Superintendent of Public Documents.

The special course for Reserve Officers occupies sixteen weeks, five recitations

[1] We are indebted to Prof. R. E. Root for the informaton regarding these courses.

per week. In this course no use is made of stereographic projections, and for text books only the publications of the Hydrographic Office mentioned above are used. Each class is given six periods on board a naval vessel in practical instruction in the use of charts and instruments.

Following is an outline of the 16 weeks' course in Navigation for Reserve Officers as recently given.

1. The assignment of time was as follows:

16 weeks, 5 recitations per week (less 13 recitation periods devoted to written tests, holidays, etc.) ..... 67 periods

Practical instruction on U. S. S. Dubuque ..... 6 periods

2. During the first nine weeks of the course, the whole class covered the following ground:

(*a*) Geometrical and trigonometrical definitions and use of logarithm tables.

(*b*) Use of chip log, patent log, sounding machine, compass, azimuth, circle pelorus, binnacle, barometers, thermometers, and log book.

(*c*) Application of variation, deviation and compass error.

(*d*) Laying courses, plotting positions and bearings, and measuring distances on chart.

(*e*) Piloting, including cross bearings, two bearings and run between, sextant angles, and use of 3-arm protractor, soundings, lights, tides, etc.

(*f*) The sailings and dead reckoning.

(*g*) Use and adjustments of sextant.

(*h*) Comparison of chronometers, and error and rate of chronometers by noon "tick."

(*i*) Navigational and astronomical definitions.

(*j*) Use of Nautical Almanac.

(*k*) Conversion of arc to time, local to standard time, mean to apparent time, and finding G. M. T.

(*l*) Corrections to observed altitudes.

(*m*) Meridian altitude of sun and constant.

(*n*) Reduction to meridian (sun).

(*o*) Time sight of sun.

(*p*) Azimuth of sun and finding compass error.

3. At the end of eight weeks the class was divided on the basis of progress made; and during the 10th and 11th weeks, two separate courses were pursued:

1st, By the more backward students (about one third of the class), a review of (*c*), (*e*) and (*f*).

2d, By the more advanced students (about two thirds of the class).

(*q*) Conversion of solar to sidereal time.

(*r*) Conversion of sidereal to solar time.

(*s*) Meridian altitudes, reduction to meridian, and time sights of stars.

(*t*) Polaris sights.

4. During the 12th, 13th and 14th weeks, the whole class covered the following subjects:

(u) Finding compass error and use of Napier's diagram.

(v) Elementary practical compensation of the compass.

(w) Lines of position and chart intersections.

(x) Naval regulations and instructions on navigational subjects.

5. During the remainder of the time available, separate courses were again pursued by the two divisions of the class:

1st Division (advanced portion of the class).

(y) Method of St. Hilaire.

(z) Day's work.

2d Division (backward portion of the class).

(q), (s), (t) and (y).

6. All of the time assigned to Navigation on board the U. S. S. Dubuque was devoted to piloting, use of charts, use of compass, sextant, etc., and much of the regular section room time was devoted to practical instruction in use of charts, noon "tick," chronometers, barometers, compass, pelorus, three-arm protractors, tide tables, deviascope, etc.

NOTES AND NEWS.

EDITED BY D. A. ROTHROCK, Indiana University, Bloomington, Indiana.

Dr. S. D. ZELDIN of the College of Hawaii has been appointed professor of mathematics in Olivet College.

Professor M. E. GRABER has been elected to the chair of mathematics in Heidelberg University, Tiffin, Ohio.

Professor C. A. BARNHART, formerly of Carthage College, has been appointed professor of mathematics in the University of New Mexico.

Dr. W. O. MENDENHALL, professor of mathematics in Earlham College, has been elected to the presidency of Friends University, Wichita, Kansas.

Assistant professor C. W. WATKEYS has been advanced to a professorship of mathematics in the University of Rochester.

Professor ARNOLD DRESDEN of the University of Wisconsin sailed for France in September for service in the Red Cross.

Dr. F. R. MORRIS has been appointed instructor in mathematics at the University of California.

Mr. H. LYLE SMITH, instructor in mathematics at Princeton University for the past two years, is now in the office of Major F. R. MOULTON of the Ordnance Department at Washington.

Professor S. L. BOOTHROYD, of the department of mathematics and astronomy, University of Washington, took an active part in the Crocker expedition for observation on the solar eclipse the past summer.

Dr. F. D. MURNAGHAN, of Rice Institute, has been appointed associate in applied mathematics at Johns Hopkins University.

Dr. ANNA PELL, associate professor of mathematics at Mt. Holyoke College, has accepted an associate professorship in mathematics at Bryn Mawr College.

Mr. C. A. HUTCHINSON, instructor in mathematics at Wittenberg College, Springfield, Ohio, has accepted an instructorship in mathematics in the College of Engineering of the University of Colorado.

Dr. THOMAS BUCK, assistant professor of mathematics at the University of California, has been commissioned a first lieutenant in the ordnance department of the army, and will be located in Washington doing research work in ballistics.

At the University of Minnesota, Associate Professor R. R. SHUMWAY has been appointed assistant dean of the College of Science, Literature and Arts; and Associate Professor W. H. BUSSEY has been appointed chairman of the department of mathematics for the year 1918–19, and also executive secretary of the School of Chemistry.

The Library Sub-Committee of the Mathematical Association of England has decided to issue a suggestive list of books suitable for mathematical libraries of secondary schools. It is proposed to divide the list into two parts, a Teachers' Library, and a Pupils' Library. A preliminary list of books on algebra and analysis, containing one hundred and sixteen titles, has been issued; it appears as a supplement to the May (1918) number of *The Mathematical Gazette*.

In the recently announced untimely death of Professor MAXIME BÔCHER American mathematicians have lost from their number one of an eminent group of men who have been in the center of American mathematical activity for a quarter century. The most noteworthy recent publication by Professor BÔCHER is his *Leçons sur les Méthodes de Sturm*, a volume which appeared last year as a monograph in the notable series edited by ÉMILE BOREL. It contains an exposition of the author's lectures delivered at the Sorbonne from November, 1913, to January, 1914. It deals with the methods of Sturm in the theory of linear differential equations and their modern developments. In it the author gives an introductory exposition of the range of ideas which have been in the center of his research activity for about twenty-five years; he includes some new results along with his account of previous work of his own and others. This book, together with Professor BÔCHER's address before the fifth international congress of mathematicians at Cambridge, affords the reader a valuable and convenient means of initiating himself into the principal contributions of their author to mathematical science. The writings of few men afford a more suitable introduction to the life-work of their author. Somewhat outside of the main current

of ideas represented in the *Méthodes de Sturm* but nevertheless connected with them is the first work through which Professor BÔCHER attracted attention, namely, his *Reihenentwickelungen der Potentialtheorie*. This appeared first in 1891 as a Göttingen dissertation of seventy pages and was later expanded into a book of 266 pages and published in 1894. Not long after the appearance of this book Professor BÔCHER began to publish his contributions in connection with the work of Sturm. Since that time his main researches have had to do with the interests thus initiated. While he has done work in other directions, it is in this and in the book of 1894 that one must seek his more important contributions.

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THIRD SUMMER MEETING OF THE MATHEMATICAL ASSOCIATION OF AMERICA.

The Association held its third summer meeting by invitation of Dartmouth College at Hanover, N. H., on Thursday, Friday and Saturday, September 5–7, 1918, in conjunction with and following the summer meeting of the American Mathematical Society. Seventy-one persons attended the sessions, including the following 42 members of the Association:

H. L. AGARD, Williams College.
R. C. ARCHIBALD, Brown University.
R. D. BEETLE, Dartmouth College.
G. D. BIRKHOFF, Harvard University.
DANIEL BUCHANAN, Queen's University.
W. D. CAIRNS, Oberlin College.
W. B. CARVER, Cornell University.
JULIA T. COLPITTS, Iowa State College.
LENNIE P. COPELAND, Wellesley College.
LOUISE D. CUMMINGS, Vassar College.
C. H. CURRIER, Brown University.
E. L. DODD, University of Texas.
C. H. FORSYTH, Dartmouth College.
A. S. GALE, University of Rochester.
O. E. GLENN, University of Pennsylvania.
C. F. GUMMER, Queen's University.
J. G. HARDY, Williams College.
E. V. HUNTINGTON, Harvard University.
W. W. JOHNSON, U. S. Navy.
FLORENCE P. LEWIS, Goucher College.
J. MATHESON, Queen's University.
HELEN A. MERRILL, Wellesley College.
F. M. MORGAN, Dartmouth College.
G. D. OLDS, Amherst College.
H. L. OLSON, New Hampshire College.
F. W. OWENS, Cornell University.
ANNA H. PALMIÉ, College for Women, Western Reserve University.
A. D. PITCHER, Adelbert College.
JESSIE G. QUIGLEY, College of Saint Teresa.
L. H. RICE, Tufts College.
R. G. D. RICHARDSON, Brown University.
E. D. ROE, JR., Syracuse University.
CLARA E. SMITH, Wellesley College.
SARAH E. SMITH, Mount Holyoke College.
H. W. TYLER, Massachusetts Institute of Technology.
OSWALD VEBLEN, Princeton University.
C. A. WALDO, Washington University.
A. G. WEBSTER, Clark University.
J. K. WHITTEMORE, Yale University.
C. B. WILLIAMS, Kalamazoo College.
F. N. WILLSON, Princeton University.
J. W. YOUNG, Dartmouth College.

The increase in railroad rates had its very evident effect in the somewhat reduced attendance as compared with the preceding summer; there were present